Spring 1999

(Real) Linear and Inner Product Spaces

Linear Space (vector space)

Definition: Let X be a non-empty set such that x and y in X implies x+y X , and α R^1 and x X implies αx X. If also one has the following properties then X is called a linear space.

1. x + y = y + x
2. x + (y + z) = (x + y) + z
3. There is a unique element 0 X such that x + 0 = x for each x X.
4. For each x there is a unique element, called (-x), such that x + (-x) = 0.
5. $\alpha (x + y) = \alpha x + \alpha y$
6. $(\alpha + \beta) x = \alpha x + \beta x$
7. $\alpha (\beta x) = (\alpha \beta) x$
8. 1 x = x
9. 0 x = 0

Inner Product Space

Definition : A linear space X is called an inner product space if there is a real valued function < , > defined on XxX (< x , y > is called the inner product of x and y) satisfying

1. $< x + y , z > = < x , z > + < y , z >$,
2. $< x , y > = < y , x >$,
3. $< \alpha x , y > = \alpha < x , y >$, and
4. $< x , x > \geq 0$ and $< x , x > = 0$ if and only if x = 0.

It can be shown that X is a normed linear space with $\| x \|^2 = < x , x >$. If the space is complete in this norm then X is called a Hilbert space.

Problems.

1) Prove that in a linear space, the vector (-x) of 4 is (-1)x.

2) Prove that the set X of polynomials of degree no more than n is a linear space by defining vector addition, scalar multiplication and verifying all the required properties.

3) Prove that the space in 2) is an inner product space if for two polynomials p and q we define

$$< p , q > = \int_0^1 p(t)\, q(t)\, dt.$$

4) Give a basis for the space in 2) and in the case n = 3 use Gram-Schmidt to obtain an orthonormal basis.