Ma 6266
Probability Distributions
Spring 1999
χ^2, t, and F

Common probability distributions and their properties can be found in many sources. Here are a few.

Ferguson, T. S. *Introduction to Mathematical Statistics.*
Hogg,R. and Craig,A. *Introduction to Mathematical Statistics.*
Rao,C.R. *Linear Statistical Inference.*

1. If X has probability density

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\,\beta^{\alpha}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

then X has a gamma distribution with parameters α and β and we write for short $X \sim G(\alpha, \beta)$. Here $\alpha > 0$ and $\beta > 0$ are arbitrary numbers.

a) Show that X has moment generating function $M(t) = E[e^{tX}] = (1 - \beta t)^{-\alpha}$ for $t < 1/\beta$.

b) Show that if $X_1, \dots, X_n$ are independent with $X_i \sim G(\alpha_i, \beta)$ then $Y = \sum_{i=1}^{n} X_i \sim G(\sum_{i=1}^{n} \alpha_i, \beta)$.

c) Show that $E[X] = \alpha\beta$.

d) Show that $Var(X) = \alpha\beta^2$.

2. The χ^2 distributions form a subset of the gamma distributions. If $X \sim G(\frac{n}{2}, 2)$, where $n \geq 1$ is an integer, then we say that X has a chi square distribution with n degrees of freedom (df) and write for short $X \sim \chi^2_n$.

a) Prove that if $Z \sim N(0,1)$ then $Z^2 \sim \chi^2_1$.

b) Prove that if $X_1, \dots, X_n$ are iid $N(\mu,\sigma^2)$ then $\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_n$.

It can be shown that if $X_1, \dots, X_n$ are iid $N(\mu,\sigma^2)$ then $\bar{X}$ and $S^2 = \dfrac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{(n-1)}$ are independent.

c) Prove that $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. Hint : Use moment generating functions.

3. The t distributions. If $Z \sim N(0,1)$ and $W \sim \chi^2_n$ and if Z and W are independent random variables, then the random variable $\dfrac{Z}{\sqrt{W/n}}$ has a t-distribution with n degrees of freedom (df).

a) One can use the usual transformation techniques to prove that the probability density function of a t random variable with n df is

$$f(x) = \frac{\Gamma(\frac{n+1}{2})\,(1 + x^2/n)^{-(n+1)/2}}{\Gamma(\frac{n}{2})\,\sqrt{n\pi}} \qquad \text{for } -\infty < x < +\infty \quad .$$

b) Prove that is if $X_1, \ldots, X_n$ are iid $N(\mu,\sigma^2)$ then $\frac{\bar{X} - \mu}{S/\sqrt{n}}$ has a t distribution with n-1 df.

4. The F distributions. If $U \sim \chi^2_m$ and $W \sim \chi^2_n$ and if U and W are independent then the random variable $\frac{U/m}{W/n}$ has an F distribution with m df for the numerator and n df for the denominator.

a) Using the usual transformation methods one can prove that the probability density of an $F_{m,n}$ random variable is

$$f(x) = \begin{cases} \dfrac{m^{m/2}\,\Gamma(\frac{m+n}{2})\,x^{m/2-1}}{n^{m/2}\Gamma(\frac{n}{2})\,\Gamma(\frac{m}{2})\,[\,1 + \frac{mx}{n}]^{(m+n)/2}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad .$$

b) Prove that if $W \sim t_n$, then $W^2 \sim F_{1,n}$.