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#9 - Find the form of the best critical region based on a sample of size n for testing

$$\text{Ho: } \theta = \theta_0 \text{ vs. Ha: } \theta = \theta_1$$

if $\theta_1 < \theta_0$ and $f(x \mid \theta) = \theta e^{-\theta x} I_{(0,\infty)}(x)$.

Solution: $r(x_1, \ldots, x_n) = (\theta_1 / \theta_0)^n \exp\{-(\theta_1 - \theta_0) \sum_{j=1}^{n} x_j\} > k$ if and only if

$(\theta_0 - \theta_1) \sum_{j=1}^{n} x_j > k'$ and this is true iff, in light of $\theta_1 < \theta_0$, $\sum_{j=1}^{n} x_j > C$.

#10 - Determine the nature of the best critical region for testing

$$\text{Ho: } \theta = \theta_0 \text{ vs. Ha: } \theta = \theta_1$$

if

$$f(x \mid \theta) = c(\theta)\, h(x)\, e^{a(\theta)b(x)}.$$

Solution:

$r(x_1, \ldots, x_n) = (c(\theta_1)/c(\theta_0))^n \exp\{ (a(\theta_1) - a(\theta_0))[\sum_{j=1}^{n} b(x_j)] > k$ if and only if

$$(a(\theta_1) - a(\theta_0))[\sum_{j=1}^{n} b(x_j)] > k'.$$

The form of the region depends upon the sign of $a(\theta_1) - a(\theta_0)$. If it is positive then the rejection region is, for some constant C, $\sum_{j=1}^{n} b(x_j) > C$ while if it is negative then the rejection region is $\sum_{j=1}^{n} b(x_j) < D$ for some constant D

 # 27 - If $X_1, \ldots, X_4$ are iid $N(\mu,1)$ and the rejection region for testing Ho: $\mu = 0$ vs. Ha: $\mu \neq 0$ rejects Ho if $|\tau| > z_{\alpha/2}$, calculate the power function's values at $\mu = 0, \pm 1/2, \pm 1$, and ± 2. Here

$$\tau = \frac{\bar{X}}{1/\sqrt{4}}$$

and the author has specified $\alpha = 0.05$.
Graph the power function $P(\mu)$, connecting the points above with a smooth curve.
Solution:

$$P(\mu) = P[\,|\tau| > z_{\alpha/2} \mid \mu\,] = 1 - P[-z_{\alpha/2} < \tau < z_{\alpha/2} \mid \mu\,]$$

$= 1 - P[-z_{\alpha/2} < 2\bar{X} < z_{\alpha/2} \mid \mu\,]$

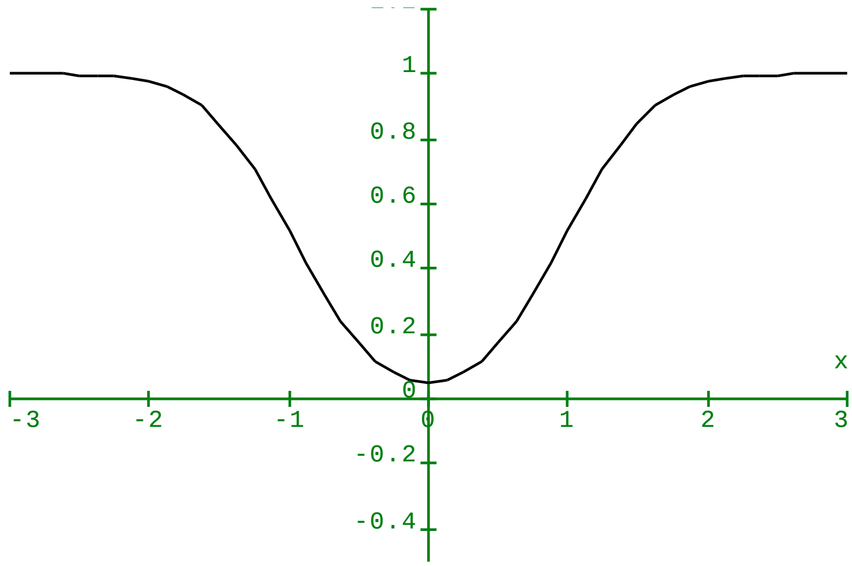
$= 1 - P[-z_{\alpha/2} < 2(\bar{X} - \mu + \mu) < z_{\alpha/2} \mid \mu\,]$
$= 1 - P[-z_{\alpha/2} < Z + 2\mu < z_{\alpha/2} \mid \mu\,]$
$= 1 - [\Phi(z_{\alpha/2} - 2\mu) - \Phi(-z_{\alpha/2} - 2\mu)]$.
Therefore $P(0) = 1 - [\Phi(z_{\alpha/2}) - \Phi(-z_{\alpha/2})] = 1 - [1 - \alpha] = \alpha$ (this confirms that α is the size of the test),
$P(1/2) = 1 - [\Phi(z_{\alpha/2} - 1) - \Phi(-z_{\alpha/2} - 1)] = 1 - [\Phi(1.96 - 1) - \Phi(-1.96 - 1)] = 1 - (.8315 - .0015) = 0.17$,
and so on. Noting that

$$\frac{dP(\mu)}{d\mu} = 2[\,\varphi(z_{\alpha/2} - 2\mu) - \varphi(-z_{\alpha/2} - 2\mu)\,]$$

we see, upon expanding the expressions in the standard normal densities and factoring out a common factor, that

$$P'(\mu) = [\, e^{2\mu z_{\alpha/2}} - e^{-2\mu z_{\alpha/2}} \,] \exp\{ \frac{-1}{2} (z^2_{\alpha/2} + 4\mu^2) \} .$$

For $\mu > 0$ this is > 0 while for $\mu < 0$ this is < 0. This shows that the following graph, generated from Maple is representative of the graph.



 # 33 - Show that the test in problem 27 is not UMP for Ho: $\mu = 0$ vs. Ha: $\mu \neq 0$.

Solution:

Working out the details, one can show that the Neyman-Pearson best test for H'o: $\mu = \mu_0$ vs. H'a: $\mu = \mu_1$, where $\mu_1 > \mu_0$, rejects H'o if

$$\frac{\bar{X} - \mu_0}{1/\sqrt{4}}$$

is too large. In our case $\mu_0 = 0$ and for size $\alpha = 0.05$, H'o is rejected if $\tau > z_{.05} = 1.645$. (In fact this test is UMP for testing H"o: $\mu = 0$ vs. H"a: $\mu > 0$ since the most powerful critical region for H'o: $\mu = \mu_0$ vs. H'a: $\mu = \mu_1$ does not depend upon the particular value of μ_1, as long as it is greater than μ_0.) This test has size 0.05 for testing Ho: $\mu = 0$ vs. Ha: $\mu \neq 0$ also. Let us compute its power at $\mu = 1/2$. We find that its power is

$$P[\tau > z_{.05} \mid \mu = 1/2] = P[\, 2\bar{X} > z_{.05} \mid \mu = 1/2] = P[Z > z_{.05} - 2\mu] = 1 - \Phi(1.645 - 1) = 0.2594.$$

The important thing here is that this is greater than P(1/2) = 0.17 which we computed for the test in #27, so the test there can not be uniformly most powerful; here is a test of the same size which has a greater power at this alternative. Of course, the test we have come up with here is terrible for testing Ho: $\mu = 0$ vs. Ha: $\mu \neq 0$ since its power for $\mu < 0$ is less than 0.05.

#s 52 and 57. These two problems illustrate that likelihood ratio tests can be complicated even when the tests can be expressed in terms of random variables whose distributions are familiar. The complication in these problems results from the critical region having no simple description; one would need a computer in the general case to come up with the precise critical region of a specified size. The following problem is different from both but is similar too and illustrates the problems involved.

Problem. Construct a likelihood ratio test of Ho: $\sigma = \sigma_0$ vs. Ha: $\sigma \neq \sigma_0$ for a normal variable if μ is unknown.

Solution.

Although it is not clear from the question, we shall assume the data consists of n iid observations $X_1, \ldots, X_n$ from a $N(\mu,\sigma^2)$ distribution. The joint density of the observations is

$$f(x \mid (\mu,\sigma)) = (\sigma\sqrt{2\pi})^{-n} \exp\{ -\frac{1}{2\sigma^2} \sum_{j=1}^{n} (x_j - \mu)^2 \}.$$

The parameter space is $\Theta = \{ (\mu,\sigma) : -\infty < \mu < \infty,\ \sigma > 0 \}$. Setting $\sigma = \sigma_0$ in the joint density and finding that the mle of μ is then $\bar{X}$, the numerator of the lrt is

$$N(x) = (\sigma_0 \sqrt{2\pi})^{-n} \exp\{ -\frac{1}{2\sigma_0^2} \sum_{j=1}^{n} (x_j - \bar{x})^2 \}.$$

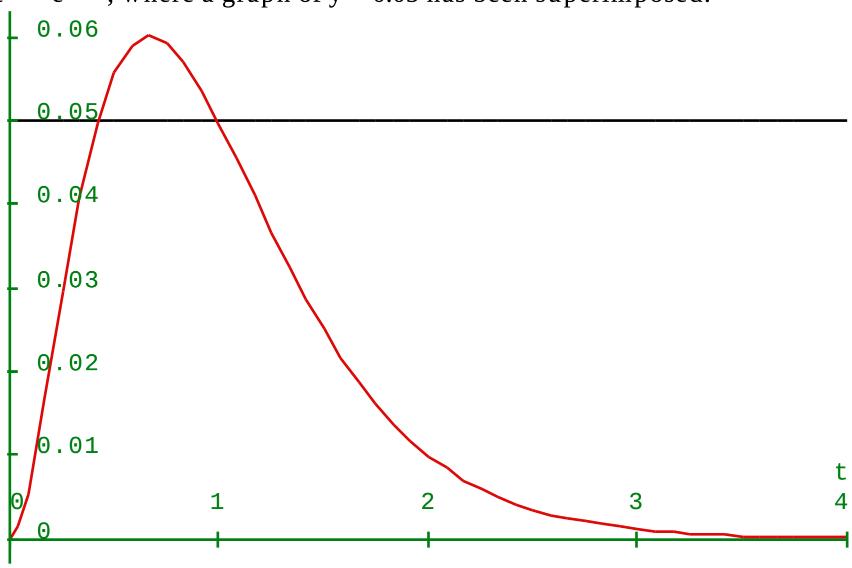
For the denominator, utilizing a previous calculation we did in class finding the mles of μ and σ (which were $(\bar{X}, \frac{1}{n} \sum_{j=1}^{n} (x_j - \bar{x})^2)$), the denominator is

$$D(x) = (\frac{1}{n} \sum_{j=1}^{n} (x_j - \bar{x})^2)^{-n/2} (\sqrt{2\pi})^{-n} \exp\{ -\frac{n}{2} \}.$$

Writing $t = t(x) = \frac{1}{n} \sum_{j=1}^{n} (x_j - \bar{x})^2$ we see that the likelihood ratio is a function of t,

$$\lambda(x) = K\ t^{n/2} e^{-at},$$

where K and a are positive constants. We seek a "solution" to $\lambda(x) < c$. Using elementary techniques from calculus for graphing functions shows that the following picture is representative of the form of these functions $t^{n/2} e^{-at}$, where a graph of y = 0.05 has been superimposed.



Therefore, for any c, the critical region of the corresponding likelihood ratio test is of the form

$$\{ t : t < r \} \quad \{ t : t > s \}$$

where r and s are the t values at which the curves intersect.

We have shown that the lrt of Ho vs Ha rejects Ho if $\sum_{j=1}^{n} (x_j - \bar{x})^2$ is either too small or too large. The precise r and s cutoffs would necessarily be determined numerically according to the size requirements. This compares with the usual (size α) test used in applications which rejects Ho if $\sum_{j=1}^{n} (x_j - \bar{x})^2 / \sigma_0^2$ exceeds $\chi^2_{n-1,\alpha/2}$ or is less than $\chi^2_{n-1,1-\alpha/2}$.

Number 57 resembles this problem except that μ is assumed known. In that case, to compute the likelihood ratio one is lead to maximize a function $c^{-n/2}e^{-a/2c}$ where $a = \sum_{j=1}^{n} (x_j - \mu)^2$. It is maximized at $c = a/n$ and in this case one winds up with a critical region like the one above except now the variable is $t(x) = \sum_{j=1}^{n} (x_j - \mu)^2$.

In problem 52 the likelihood ratio

$$\lambda(x,y) = N(x,y)/D(x,y),$$

turns out, picking up where we left off in the lecture, to have

$$D(x,y) = n^{n/2} m^{m/2} \frac{(2\pi e)^{-(n+m)/2}}{u^{n/2} v^{m/2}}$$

where $u = \sum_{j=1}^{n} (x_j - \bar{x})^2$ and $v = \sum_{j=1}^{n} (y_j - \bar{y})^2$. We have

$$N(x,y) = (2\pi)^{-(n+m)/2} \max_c c^{-(n+m)/2} e^{-(u+v)/2c}$$

$$= (n+m)^{(n+m)/2} \frac{(2\pi e)^{-(n+m)/2}}{(u+v)^{(n+m)/2}}.$$

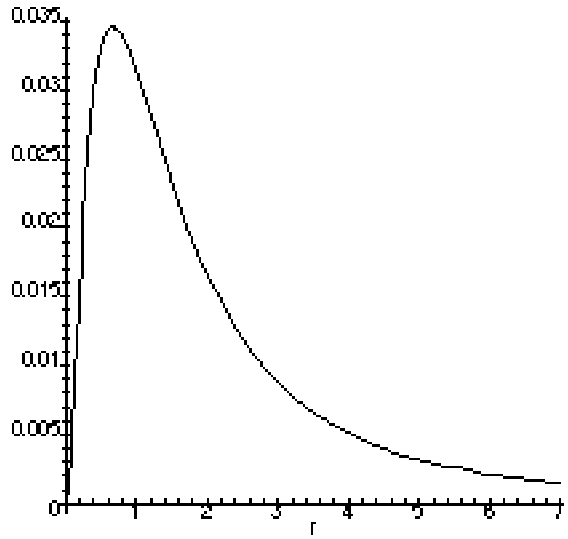
Therefore the critical region depends only on the values of u and v. Actually, it only depends upon the ratio $r = u/v$ of the two since

$$\lambda(x,y) = K \frac{u^{n/2} v^{m/2}}{(u+v)^{(n+m)/2}}$$

$$= K \frac{u^{n/2}}{(u+v)^{n/2}} \frac{v^{m/2}}{(u+v)^{m/2}}$$

$$= K \frac{r^{n/2}}{(1+r)^{(n+m)/2}}.$$

A graph of this as a function of r is given below for n = 4 and m = 6 and it is representative of the general situation; the critical region is split into two disjoint intervals whose endpoints can only be found with a computer.



Therefore, the likelihood ratio test rejects Ho: $\sigma_1 = \sigma_2$ if the ratio

$$\tau = \frac{\sum_{j=1}^{n} (x_j - \bar{x})^2 / (n-1)}{\sum_{j=1}^{n} (y_j - \bar{y})^2 / (m-1)}$$

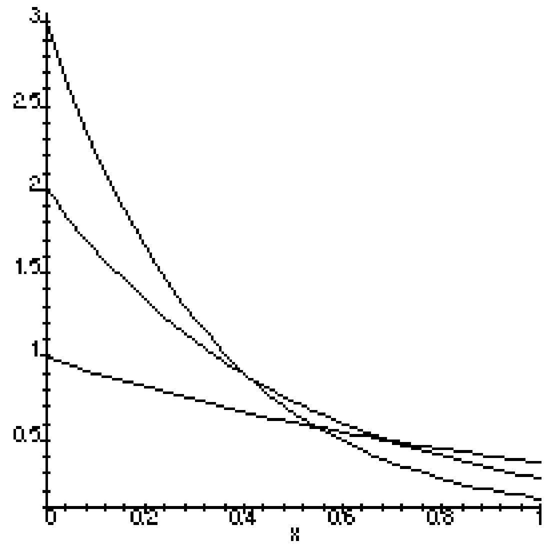
is too small or too large. The precise boundaries would require a numerical routine.

In practice one uses the test which rejects Ho if $\tau < F_{n-1,m-1,1-\alpha/2}$ or if $\tau > F_{n-1,m-1,\alpha/2}$. It is an interesting further exercise to show that if n = m then this is precisely the likelihood ratio test.

 # 69 Given three exponential densities $f(x \mid \theta) = \theta e^{-\theta x} I_{(0,\infty)}(x)$, with θ = 1, 2, and 3, find the Bayes multiple decision rule if $\pi_j = 1/3, j = 1, 2, 3$ and a single observation is to be taken. (We assume zero-one loss although the authors do not state that.)

Solution:

With equal prior probabilities we know that the solution chooses that j for which $f(x \mid \theta_j)$ is maximum since the posteriors are proportional in that case. Plots of these three are below.



The intersection points, in terms of s, are found to be $1/2, 1/\sqrt{3}$, and $2/3$ by setting $s = e^{-x}$ and solving $s = 3s^3$, $s = 2s^2$, and $2s^2 = 3s^3$. Therefore, in terms of x, they are $\ln 2$, $(\ln 3)/2$, and $\ln(3/2)$ and the Bayes rule selects population 3 if $X \le \ln(3/2)$, population 2 if $\ln(3/2) < X < \ln 2$, and population 1 if $X \ge \ln 2$.