Mortgage Notes

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1 Plain-vanilla mortgages

What is the (constant) monthly payment C on a 6.5% ($\alpha = 0.065$) 30 year plain vanilla mortgage for the amount P_0? At the end of the k^{th} payment, assuming no early payments of principal, what is the equity E_k in the property? To answer these questions we shall assume that the principal is P_0 and that the interest rate α refers to the payment calculated on the unpaid portion of the principal. Thus if the balance of the principal remaining after payment n is B_n then the interest portion of the $n+1^{st}$ payment is $I_{n+1} = \frac{\alpha}{12}B_n$ and the portion $D_{n+1} = C - I_{n+1}$ is the portion which goes toward reducing the principal. The equity after n payments is therefore

$$E_n = D_1 + D_2 + \dots D_n$$

and the balance remaining is $B_n = P_0 - E_n$. Under the assumptions, setting $B_0 = P_0$, one has for $n \geq 1$

$$B_n = \gamma B_{n-1} - C,$$

where $\gamma = 1 + \alpha/12$. This is a consequence of the fact that

$$I_n = \frac{\alpha}{12}(P_0 - D_1 - \dots D_{n-1}) = \frac{\alpha}{12}B_{n-1}$$

and

$$\begin{aligned} B_n &= P_0 - E_n \\ &= P_0 - E_{n-1} - D_n \\ &= P_0 - E_{n-1} - (C - I_n) \\ &= B_{n-1} - C + \frac{\alpha}{12}B_{n-1} \\ &= (1 + \frac{\alpha}{12})B_{n-1} - C. \end{aligned}$$

Consider solving the difference equation

$$B_n - \gamma B_{n-1} + C = 0. \tag{1}$$

We seek a solution in the form $B_n = U\lambda^n + V$ and find that in this case

$$U\lambda^{n-1}(\lambda - \gamma) + V - \gamma V + C = 0.$$

Thus for any U choosing $\lambda = \gamma$ and $V = \frac{C}{\gamma-1}$ yields a solution to (1). The solution we seek has the boundary condition $B_N = 0$, where N is the number of payments (12×30 for a thirty year plain vanilla mortgage). Thus since

$$B_n = U\gamma^n + \frac{C}{\gamma - 1}$$

one has $U = -\frac{C}{\gamma-1}\gamma^{-N}$ so that the unpaid balance on the principal after the n^{th} payment is

$$B_n = \frac{C}{\gamma - 1}(1 - \gamma^{n-N}). \tag{2}$$

Since we know

$$\begin{aligned} C &= D_1 + I_1 \\ &= P_0 - B_1 + I_1 \\ &= P_0 - \frac{C}{\gamma - 1}(1 - \gamma^{1-N}) + \frac{\alpha}{12}P_0 \\ &= \gamma P_0 - \frac{C}{\gamma - 1}(1 - \gamma^{1-N}), \end{aligned}$$

we have, solving the latter for C that

$$C[1 + \frac{1 - \gamma^{1-N}}{\gamma - 1}] = \gamma P_0,$$

or

$$\frac{C}{\gamma - 1}\gamma(1 - \gamma^{-N}) = \gamma P_0.$$

It follows that the fixed payment amounts are

$$C = \frac{\gamma - 1}{1 - \gamma^{-N}}P_0 \tag{3}$$

For our $\alpha \times 100\%$ 30 year mortgage this is

$$C = \frac{\alpha}{12}\frac{(1 + \frac{\alpha}{12})^{360}}{(1 + \frac{\alpha}{12})^{360} - 1}P_0.$$

Combining (2) and (3) one has the expression of the equity after the n^{th} payment as

$$E_n = P_0 - B_n = P_0 - \frac{1 - \gamma^{n-N}}{1 - \gamma^{-N}}P_0 = \frac{\gamma^n - 1}{\gamma^N - 1}P_0.$$

One can employ the calculator in the finance section at URL: *http://www.opti-stat.net* to compute answers to our opening questions. The monthly payment on an amount $P_0 = \$200,000$ is $C = \$1,264.14$ while, for example, the equity at 5 years is $E_{60} = \$12,778.05$.