

Block Voting

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1 Introduction

Once upon a time there was a department head who routinely rounded up his relatively small band of cohorts before each faculty meeting to make sure they were in attendance. The group carried the day frequently and were able to run the department as they wished. Perhaps counter-intuitively, only a relatively small cohesive group (on the order of the square root of the department size) is needed for this to result in their frequent success. A group of 10, for example, will prevail around 90% of the time in a department of 54. The desires of a group of just $n = 5$ will be realized around 80% of the time in a department of $N = 30$ members.

2 Details

Suppose a committee consists of N persons of which $n < N$ vote as a block. Of the remainder assume that they vote independently as $B(1, 1/2)$; that is, on any randomly appearing issue they are roughly split down the middle. What is the probability $W(N, n)$ the block prevails in its opinion over time?

Let X_1, X_2, \dots, X_{N-n} be iid $B(1, 1/2)$. Then taking $Y = \sum_{j=1}^{N-n} X_j$ one has $W(N, n) = P[n + Y > N/2]$ for simple majority voting. Clearly, if $n > N/2$ then the block prevails in all votes. Otherwise one has the Table below for small N, n and the large sample approximation that follows.

2.1 Large committees

Let $n/N = c \in (0, 1)$ and assume N is large. Then

$$\begin{aligned} W(N, n) &= P[Y > N/2 - n] \\ &= P[Y > N(1/2 - c)] \\ &= P\left[\frac{Y}{N-n} > \frac{N(1/2 - c)}{N(1-c)}\right] \\ &= P\left[\frac{Y}{N-n} - 1/2 > \frac{1/2 - c}{1-c} - 1/2\right] \\ &= P\left[\frac{Y}{N-n} - 1/2 > -c/2\right] \end{aligned}$$

$$\begin{aligned}
&= P[2\sqrt{N(1-c)}(Y/N(1-c) - 1/2) > -\sqrt{N(1-c)c}] \\
&\rightarrow P[Z > -\infty] \\
&= 1
\end{aligned}$$

In fact, for $c_N = a/\sqrt{N}$ one has

$$W(N, n) \rightarrow 1 - \Phi(-a) > 1/2.$$

For example, using the approximation, if $a = 2$ then in a committee of 49 if $n = 49 \cdot 2/\sqrt{49} = 14$ members vote as a block, they will prevail around 98% of the time (compare with the exact value of 0.986 for 14 from a department of size 50 from the tabled values).

In a large population like the US's 174,136,341 members between the ages of 18 and 64, roughly the voting age population, a group voting solidly as a block as small as $n = 2 \times 13196.07294 \approx 27,000$ (in a system of strictly popular voting) would prevail around 98% of the time when the remainder of the population was roughly evenly split over the candidates!

2.2 Finite committees

Let $Y \sim B(N - n, 1/2)$ and simply look up or calculate $P[Y > N/2 - n]$. In a committee of $N = 5$ members, for example, in which $n = 2$ vote together, the group will prevail around

$$P[Y > 1/2] = 1 - P[Y = 0] = 1 - \binom{3}{0} 2^{-3} = 0.875$$

of the time. Values for committee sizes beginning at $N = 8$ and as large as 54 are tabled below.

$N/2, n$	2	3	4	5	6	7	8	9	10	11	12	13	14
4	0.656	0.813											
5	0.637	0.773	0.891										
6	0.623	0.746	0.855	0.938									
7	0.613	0.726	0.828	0.910	0.965								
8	0.605	0.709	0.806	0.887	0.945	0.980							
9	0.598	0.696	0.788	0.867	0.927	0.967	0.989						
10	0.593	0.685	0.773	0.849	0.910	0.954	0.981	0.994					
11	0.588	0.676	0.760	0.834	0.895	0.941	0.971	0.989	0.997				
12	0.584	0.668	0.748	0.820	0.881	0.928	0.962	0.982	0.994	0.998			
13	0.581	0.661	0.738	0.808	0.868	0.916	0.952	0.975	0.989	0.996	0.999		
14	0.577	0.655	0.729	0.798	0.857	0.905	0.942	0.968	0.985	0.994	0.998	1.000	
15	0.575	0.649	0.721	0.788	0.846	0.895	0.933	0.961	0.979	0.990	0.996	0.999	1.000
16	0.572	0.644	0.714	0.779	0.837	0.885	0.924	0.953	0.974	0.987	0.994	0.998	0.999
17	0.570	0.640	0.708	0.771	0.828	0.876	0.916	0.946	0.968	0.983	0.992	0.996	0.999
18	0.568	0.636	0.702	0.763	0.819	0.868	0.908	0.939	0.962	0.978	0.989	0.995	0.998
19	0.566	0.632	0.696	0.757	0.811	0.859	0.900	0.932	0.956	0.974	0.986	0.993	0.997
20	0.564	0.629	0.691	0.750	0.804	0.852	0.892	0.925	0.951	0.969	0.982	0.990	0.995
21	0.563	0.625	0.686	0.744	0.797	0.845	0.885	0.919	0.945	0.965	0.979	0.988	0.994
22	0.561	0.622	0.682	0.739	0.791	0.838	0.879	0.912	0.939	0.960	0.975	0.985	0.992
23	0.560	0.620	0.678	0.734	0.785	0.832	0.872	0.906	0.934	0.955	0.971	0.982	0.990
24	0.559	0.617	0.674	0.729	0.780	0.826	0.866	0.900	0.928	0.951	0.967	0.980	0.988
25	0.557	0.615	0.671	0.724	0.774	0.820	0.860	0.894	0.923	0.946	0.964	0.976	0.986
26	0.556	0.612	0.667	0.720	0.769	0.814	0.854	0.889	0.918	0.941	0.960	0.973	0.983
27	0.555	0.610	0.664	0.716	0.765	0.809	0.849	0.884	0.913	0.937	0.956	0.970	0.981

Table 1: Proportion of time block of n prevails in a committee of size N .