L_1 Scaling Functions and Absolute Continuity of Random Expansions

by

A. Deliu
University of South Carolina

and

M. C. Spruill
Georgia Institute of Technology

Abstract

There is a non-trivial integrable solution to a two scale difference equation with non-negative coefficients if and only if the corresponding random expansion is absolutely continuous. In this setting, some new sufficient conditions and some new necessary conditions are given which enable the complete and simple characterization of all two scale equations, with no more than six coefficients, possessing an integrable solution.

AMS Subject Classification : Primary- 39B22, 42A38, Secondary 26A46, 60J05, 26A30, 26C10

Key Words: refinement equation, dilation equation, characteristic function

1. Introduction.

The random k-adic, or simply random, expansions of this study are

(RE) $$Y = \sum_{n\geq 1} C_n k^{-n},$$

where $C_1, C_2, \ldots$ are independent random variables distributed as C, $P[\, C = j \,] = c_j / k$ for $j = 0, 1, \ldots, N$, $c_j \geq 0$, $\sum_{j=0}^{N} c_j = k$ is an integer greater than 1, and, to avoid trivial cases, at least two of the c_i's are non zero. Our goal is the simple description of conditions under which Y is absolutely continuous.

If $N \leq k-1$ and $\nu(\mathbf{p})$, $\mathbf{p} = \frac{1}{k}(\, c_0, \ldots, c_N)$, is the Borel measure on $\Omega = [0,1]$ induced by the corresponding random variable Y, then all the measures $\nu(\mathbf{p})$ are mutually singular; only $\mathbf{p} = \frac{1}{N+1}\mathbf{1}$ is absolutely continuous with respect to Lebesgue measure. Mutual singularity of $\nu(\mathbf{p})$ and $\nu(\mathbf{p}')$, with $c'_0 \neq c_0$ for example, is a consequence of Kolmogorov's strong law applied to $\frac{1}{j} N(j,\omega)$, where $N(j,\omega) = \#\{\, C_i(\omega) = 0 : i \leq j \,\}$. Since $\frac{1}{j} N(j,\omega)$ converges a.s. to c_0 / k or to c'_0 / k according to which of $\mathbf{p}$ or $\mathbf{p}'$ holds, $\nu(\mathbf{p})$ and $\nu(\mathbf{p}')$ place their masses on essentially disjoint sets. When $N > k-1$, much less is known and this is the condition under which our investigations of absolute continuity are conducted below.

The characteristic function of Y is

(C) $$E[\, \exp\{itY\} \,] = \varphi_Y(t) = \prod_{n\geq 1} p(\exp\{itk^{-n}\}),$$

where p is the polynomial given by

(P) $$p(z) = \frac{1}{k} \sum_{n=0}^{N} c_n z^n ,$$

and although it is clear that characterization of the absolute continuity of Y must be possible in terms of the polynomial p, this ultimate goal is attained here in only a small number of completely general cases.

The cumulative distribution function F of Y must satisfy

(CD) $$F(x) = \frac{1}{k} \sum_{n=0}^{N} c_i\, F(kx - n),$$

and since the solution of equation (CD) in the space BV of functions of bounded variation will be shown to be unique up to scalar multiplication, the paper of Dubins and Freedman (1966) shows that the measure corresponding to F must have no discrete spectrum and is either purely singular continuous or purely absolutely continuous. The only non-trivial solutions to (CD) in BV are (up to sign) cdfs, so it also follows that absolute continuity of Y is equivalent to the existence of a non-trivial L_1 solution to the two-scale difference equation

(D) $$f(x) = \sum_{n=0}^{N} c_n f(kx - n)$$

whose solution is called a scaling function and which, when it exists, can be taken to be the density of Y.

When does the equation (D) possess a non-trivial solution in L_1 ? For example, when N=5 and k=2 the only absolutely continuous solutions are for the following three situations: (i) $c_0 + c_2 + c_4 = c_1 + c_3 + c_5$, (ii) $c_1 - c_3 + c_5 = 0 = c_0 - c_2 + c_4$, (iii) $c_3 = 0$, $c_2 = 0$, $c_1 - c_5 = 0 = c_0 - c_4$.

The general situation $\sum_j c_{2j} = \sum_j c_{2j+1}$, represented by case (i) has been studied extensively in the wavelet literature. In Micchelli and Prautzsch (1987), for example, they show in Corollary 5.1 that under this condition, when all c_i are positive, there is a unique continuous solution to (D) and, in an example, that if one or more c's are zero then continuity may fail. They also prove that the order of the zero of p at -1 dictates certain smoothness properties of the solution f (see also Strang (1989)). This is to be contrasted with the more general case, still assuming (i), when the coefficients can be of mixed signs. Here Colella and Heil (1992) characterize the c's which result in continuous solutions in terms of the joint spectral radius of two matrices and numerically investigate where continuous solutions exist. See also Colella and Heil (1994) and Berger and Wang (1992).

Recently Lau and Wang (1994) have characterized the existence of L_p solutions to (D) in terms of the spectral radius of a certain matrix. Still, the actual determination of whether or not an absolutely continuous solution exists generally depends upon numerical computations. We do not know whether their technique can be used for non-negative coefficients to characterize simply those polynomials p for which an absolutely continuous solution exists. The largest example treated in their paper for L_1 is four coefficients and that is carried out only in the case $\sum_j c_{2j} = \sum_j c_{2j+1}$.

Our main results utilize the characteristic function and the primary tools employed are the Riemann-Lebesgue lemma and an operator T which arose for us as an averaging operator and which is related to an operator employed by T. Eirola (1992) in his study of smoothness of solutions to dilation equations (see also Heil (1992) and Conze and Raugi (1990)). Using the Riemann-Lebesgue lemma it is shown that if F is absolutely continuous then there must be certain special roots of p on the unit circle in the complex plane which form a "blocking set" of vertices in a certain forest of directed trees. This general result shows immediately that absolutely continuous solutions of (CD) are extremely rare. Using the operators T the following simple sufficient condition, for example, can be given for the existence of an L^1 solution to (D): if $\sum_{j=0}^{k-1} \left| p(e^{2\pi i (\frac{x+j}{k})}) \right| \leq 1$ for all x [0,1] then (D) has a non-trivial L_1 solution.

2. Solutions to (CD).

By a solution to (CD) is meant a function $F = F^+ - F^-$ in BV. In this section it is proven that (CD) always has a unique solution F which is an honest cumulative distribution function (cdf) of the random variable Y of (RE), that F is supported on $[0, \frac{N}{k-1}]$, and that it is the limit of iterates of the operator defined by the right hand side of (CD).

Convergence of iterates of the right hand side of (D) has been investigated. See Daubechies and Lagarius (1991) for example. The iterates $f^{(m)}(x)$ starting with an initial function $f^{(0)}$ are

$$f^{(m)}(x) = \sum_{n=0}^{N} c_n f^{(m-1)}(kx - n).$$

Generally this sequence of functions may not converge and convergence depends upon the choice of the initial function. The cascade algorithm, as it is called in the wavelet literature as applied to the equation (D), is the version obtained from a particular set of starting conditions. See also Heil (1992) for a different starting f for the cascade algorithm. It will be seen that the iteration scheme using the equation (CD) always converges to the unique solution cdf F of (CD) regardless of the starting $F^{(0)}$.

Shorthand for a random variable Y having the cdf F below will be Y ~ F. Define the (CD) iteration on the space of cdf's by

$$H^{(n)}(x) = \frac{1}{k}\sum_{j=0}^{N} c_j\, H^{(n-1)}(kx - j).$$

Theorem2.1. For any cdf H, $H^{(n)} \to F$ as $n \to \infty$, convergence being pointwise at each continuity point of F, where F is the cdf of Y, and F is the unique (up to sign) non-trivial solution in BV to the equation (CD).

Proof: Let X, C_1, C_2, ... be independent random variables where the C_j's are distributed as C and X ~ H. For any random variable W, define $S_j W = (W+C_j)/k$. Observing that

$$P[\, SX \le t\,] = \frac{1}{k}\sum_{j=0}^{N} c_j\, H(\, kx - j),$$

it follows that $\prod_{j=1}^{n} S_j X \sim H^{(n)}$. Although the sequence of random variables $\prod_{j=1}^{n} S_j X$ does not converge in L_2, $X_n = \sum_{j=1}^{n} C_j\, k^{-j} + X\, k^{-n}$ have the same distributions and, by elementary calculations, clearly converge in L_2, and hence in law, to Y showing that $H^{(n)} \to F$ as $n \to \infty$, where F is the cdf of Y.

As a consequence one also has that the characteristic function of Y is given in (C), for the characteristic function φ_n of X_n satisfies $\varphi_n(t) = p(e^{it/k})\, \varphi_{n-1}(t/k)$ and with $\varphi_0(t) = E[\, e^{itX}]$ one has $\varphi_n(t) = \prod_{j=1}^{n} p(e^{itk^{-j}})\, \varphi_0(tk^{-n})$. Since $\varphi_0(t)$ is a characteristic function and is continuous at 0 with $\varphi_0(0) = 1$, one has $\lim_{n\to\infty} \varphi_n(t) = \varphi_Y(t)$.

The proceedings so far entail that for all continuity points x of F for which kx - j are also continuity points of F for all j = 0, ..., N (this is always an uncountable dense set)

$$F(x) = \lim\; H^{(n)}\,(x) = \frac{1}{k}\sum_{j=0}^{N} c_j\, \lim\; H^{(n-1)}\,(\, kx - j) = \frac{1}{k}\sum_{j=0}^{N} c_j\, F(\, kx - j),$$

so that F solves (CD) by right continuity.

To see that F is unique up to scalar multiplication in BV, let $G(x) = G^{(+)}(x) - G^{(-)}(x)$ be any non-trivial solution in BV and assume that both $G^{(\pm)} \neq 0$. Then by appropriate choice of a non-zero constant b it can be assumed that $G = G^{(+)} - b\, G^{(-)}$ solves (CD) and that $G^{(+)}$ and $G^{(-)}$ are both honest cdf's. If X (+) and X(-) are the corresponding random variables then the characteristic functions $\varphi_{\pm}$ satisfy for any $-\infty < t < +\infty$,

$$\varphi_+(t) - b\, \varphi_-(t) = p(e^{it/k})\, [\, \varphi_+(t/k) - b\, \varphi_-(t/k)\,].$$

Iterating this and taking limits yields

$$\varphi_+(t) - b\ \varphi_-(t) = \varphi_Y(t) \lim_{n\to\infty} [\varphi_+(tk^{-n}) - b\,\varphi_-(tk^{-n})] = 0,$$

so that b = 1 and $G^{(+)} = G^{(-)}$. This entails G trivial which is contrary to our assumption. Therefore, the only signed measure solving (CD) is trivial in this case. If there were another non-trivial honest cdf F*solving (CD), then choosing $G^{(+)}$ = F and $G^{(-)}$ = F* shows F = F*. It follows that the only non-trivial solutions to (CD) are non-zero scalar multiples of F, where F is an honest cdf. ◊

Henceforth, we refer to the cdf F of Y as the solution of (CD) and note that it has also been shown that the absolute continuity of Y is equivalent to the existence of an L_1 solution to (D) which, if it exists, must be of one constant sign. In conjunction with our assumption on F, this is taken to be non-negative and is also the density of Y.

From the fact that Y ~ F for the solution F of (CD) it is plain by simple bounds that with probability 1, $0 \le Y \le \frac{N}{k-1}$ so that F is compactly supported.

In terms of the notation of Dubins and Freedman (1966), the present problem has $\Omega = [0, \frac{N}{k-1}]$, $\Gamma = \{\gamma_0, \ldots, \gamma_N\}$, and $\gamma_j(\omega) = \frac{\omega+j}{k}$. By their Theorems 2.5 and 4.5 one can conclude that, in the standard decomposition $F = F_{ac} + F_s + F_d$ of the cdf solution F to (CD), $F_d = 0$ and one or the other of F_{ac} and F_s is 0 but, as they point out, "it is difficult to decide which." For an indication of the difficulty of this decision in similar circumstances, see Jesson and Wintner (1935), Kershner and Wintner (1935), Erdös (1940), Garsia (1962), and Dubins and Freedman (1966).

3. Some sufficient conditions for absolute continuity.

It is well known that if a random variable X has a characteristic function φ in $L_1(-\infty, +\infty)$ then X is absolutely continuous with a continuous density. In this section it is proven that if r ≥ 1 and condition (Tr) holds then φ L_r. Here the condition is

(Tr) for all x [0,1], $$\sum_{j=0}^{k-1} \left| p(e^{2\pi i(\frac{x+j}{k})}) \right|^r \le 1,$$

where p is the polynomial (P). Of particular interest are the cases r (1,2] and r =1, for in the former case Y is absolutely continuous (see Kawata (1972)) with a density solving (D) in L_s for s = r/(r-1), while in the latter, (D) must have a non-trivial continuous (on (-∞, +∞)) solution. Otherwise, if (Tr) holds for an integer r ≥ 2, then whenever $Y_1, \ldots, Y_r$ are iid, the sum $\sum_{j=1}^{r} Y_j$ is absolutely continuous and the corresponding convolution F^{*r} solves (CD) (see Micchelli and Prautzsch (1987)) with N ' = rN and coefficients $d = k\,(c/k)^{*r}$.

Define the linear operator T_r by

$$T_r f(x) = \sum_{j=0}^{k-1} \left| p(e^{2\pi i(\frac{x+j}{k})}) \right|^r f(\frac{x+j}{k})$$

and let Ψ(t) = | φ(2πt) |, where φ is as in (C).

Lemma 3.1. If W ~ U(0,1) then

$$\int_0^{2\pi k^m} |\,\varphi(t)\,|^r\,dt \;=\; 2\pi\,E[\,T_r^m\,\Psi^r(W)\,].$$

Proof: We use

$$\text{(3.1)} \qquad \varphi(k^m t) = \prod_{n=0}^{m-1} p(e^{itk^n})\,\varphi(t).$$

One has

$$\int_0^{2\pi k^m} |\,\varphi(t)\,|^r\,dt \;=\; 2\pi k^m\,E[\,\Psi^r(W) \prod_{n=0}^{m-1} |\,p(e^{2\pi i W k^n})\,|^r\,]$$

$$= 2\pi k^m\,E[\,\Psi^r(W) \prod_{n=0}^{m-1} |\,p(e^{2\pi i \{W k^n\}})\,|^r\,],$$

where $W \sim U(0,1)$ and $\{s\}$ is the fractional part of s. Writing $W = \sum_{j\ge 1} W(j)\,k^{-j}$, where W(1), W(2), … are independent and uniform on {0,1,…,k-1}, and conditioning on $\{kW\} = \sum_{j\ge 1} W(j+1)\,k^{-j} = y$ one has

$$\int_0^{2\pi k^m} |\,\varphi(t)\,|\,dt = 2\pi k^m\,E_y\,\{E[\,\Psi^r\,(W(1)/k + y/k)\;|\,p(e^{2\pi i(W(1)/k\,+\,y/k)})\,|^r \prod_{n=1}^{m-1} |\,p(e^{2\pi i y k^{n-1}})\,|^r\,|\;y\,]\,\}.$$

Since P[W(1) = j] =1 / k, j = 0, … ,k-1, and is independent of y,

$$\int_0^{2\pi k^m} |\,\varphi(t)\,|\,dt = 2\pi k^m\,E_y\,\{\,\frac{1}{k}\sum_{j=0}^{k-1} \Psi^r\!\left(\frac{j+y}{k}\right)\,|\,p(e^{2\pi i\left(\frac{y+j}{k}\right)})\,|^r \prod_{n=0}^{m-2} |\,q(e^{2\pi i y k^n})\,|\;\;|\;y]\,\}$$

$$= 2\pi k^{m-1}\,E[\,T_r\,\Psi^r(W) \prod_{n=0}^{m-2} |\,p(e^{2\pi i \{W k^n\}})\,|^r\,] = \ldots = 2\pi\,E[\,T_r^m\,\Psi^r(W)\,]. \qquad \Diamond$$

Theorem 3.2. If (Tr) holds for all x [0,1] then φ is in L_r.

Proof: It suffices to prove that $\int_0^\infty |\,\varphi(t)\,|^r\,dt < \infty$. Since

$$\int_0^\infty |\,\varphi(t)\,|^r\,dt = \lim_{m\to\infty} \int_0^{2\pi k^m} |\,\varphi(t)\,|^r\,dt = \lim_{m\to\infty} 2\pi\,E[\,T_r^m\,\Psi^r(W)\,] = \lim_{m\to\infty} 2\pi\,||\,T_r^m\,\Psi^r\,||_1$$

and

$$|\,T_r f(x)\,| = |\,\sum_{j=0}^{k-1} |\,p(e^{2\pi i\left(\frac{x+j}{k}\right)})\,|^r\;f\!\left(\frac{x+j}{k}\right)| \le \sum_{j=0}^{k-1} |\,p(e^{2\pi i\left(\frac{x+j}{k}\right)})\,|^r\;||\,f\,||_\infty \le ||\,f\,||_\infty$$

it follows that $||\,T_r^m f\,||_1 \le ||\,T_r^m f\,||_\infty \le ||\,f\,||_\infty$ and hence that φ L_r. ◊

Example 3.3. Consider the polynomial $\left(\frac{1+z}{2}\right)^2$. Since $p(e^{2\pi i t}) = m^2(e^{2\pi i t})$, $m(z) = (1+z)/2$ and

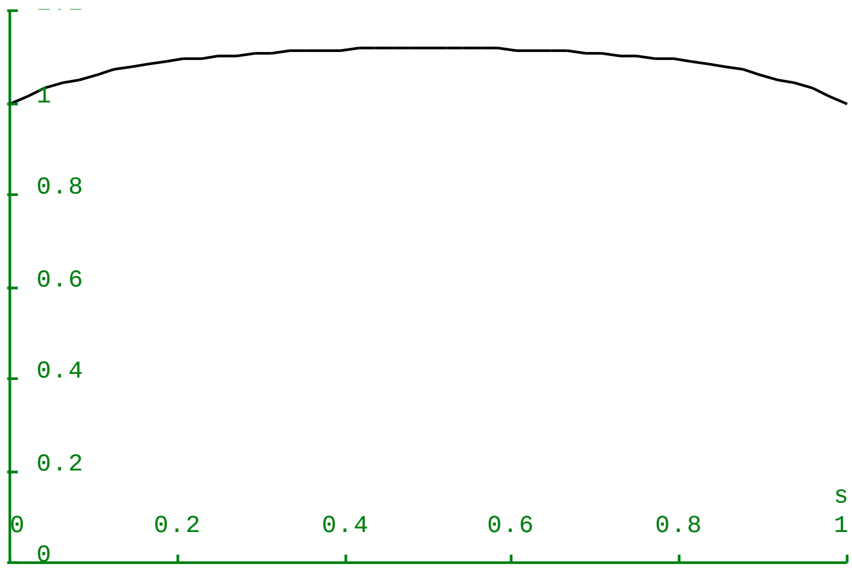
$$|\,p(e^{2\pi i t})\,| = |\,m(e^{2\pi i t})\,|^2 = \frac{1}{k^2}\,[\,\sum_{j=0}^{N} c_j^2 + \sum_{r\ne s} c_r\,c_s\,\cos(2\pi t(r-s))\,]\,.$$

Here k = 2 and $|\,p(e^{2\pi i t})\,| = (\,2 + 2\cos(2\pi t))/4 = (1+\cos(2\pi t))/2$. Therefore

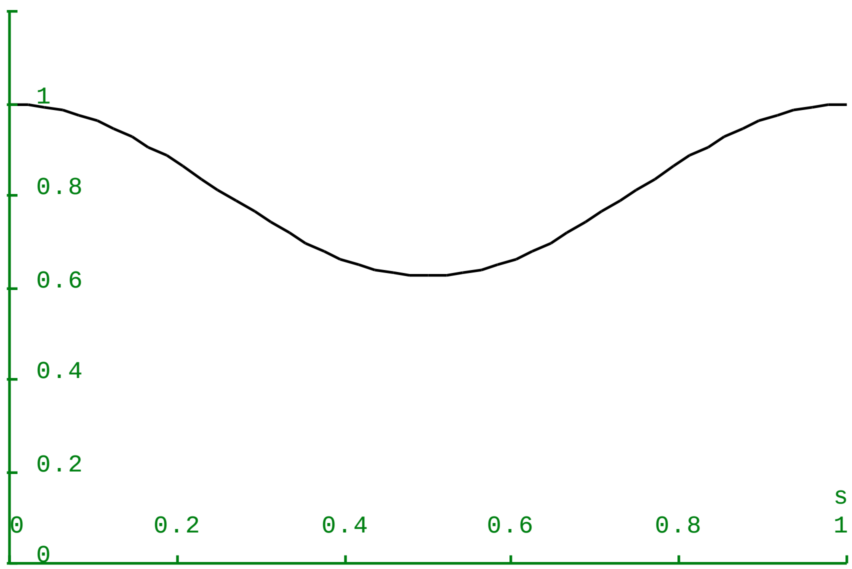
$$| p(e^{2\pi i t/2}) | + | p(e^{2\pi i (t+1)/2} | = \frac{1}{2} [2 + \cos(\pi t) + \cos(\pi t + \pi)] = 1$$

and in this example (Tr) is satisfied with r = 1. ◊

Example 3.4. Consider $p(z) = \frac{1}{2}(\alpha + z + (1-\alpha) z^2)$ with k =2. Below is typical plot of the function on the left hand side of the inequality (T1), in this case, for $\alpha = 1/4$.



The condition (T1) is not satisfied. Next is the plot of (T2), again for $\alpha = 1/4$.



This plot shows that the cdf F is absolutely continuous and that its density is in L_2 when $\alpha = 1/4$. There is, in fact, an L^2 solution to (D) for all α by the next result.

Theorem 3.5. If for all $u = 0,1,\ldots,k\text{-}1$, $\sum_j c_{kj+u} = 1$ then Y is absolutely continuous and (D) has an L^2 solution.

Proof: Assuming that $\sum_j c_{kj+u} = 1$ for all $u=1,\ldots,k\text{-}1$ it will be shown that (T2) is satisfied.

Writing $k\, p(e^{i2\pi t}) = \sum_{v=0}^{k-1} \sum_w c_{kw+v}\, e^{2\pi i t(kw+v)}$ one has

$$k^2 \sum_{j=0}^{k-1} \left| p(e^{2\pi i (\frac{s+j}{k})}) \right|^2$$

$$= \sum_{j=0}^{k-1} \sum \sum \sum \sum c_{kw(1)+v(1)}\, c_{kw(2)+v(2)}\, e^{2\pi i (s+j)(w(1)-w(2))}\, e^{2\pi i (s+j)(v(1)-v(2))/k},$$

where the sums on v's are 0 to k-1 and the sums on w's are over all integers. Then, rearranging, the right hand expression is

$$\sum \sum \sum \sum c_{kw(1)+v(1)}\, c_{kw(2)+v(2)}\, e^{2\pi i s(w(1)-w(2))}\, e^{2\pi i s(v(1)-v(2))/k} \sum_{j=0}^{k-1} e^{2\pi i j(v(1)-v(2))/k}$$

and since for $v(1) \neq v(2)$

$$\sum_{j=0}^{k-1} e^{2\pi i j(v(1)-v(2))/k} = (1-e^{2\pi i k(v(1)-v(2))/k}) \,/\, (1-e^{2\pi i (v(1)-v(2))/k}) = 0,$$ one has

$$k^2 \sum_{j=0}^{k-1} \left| p(e^{2\pi i (\frac{s+j}{k})}) \right|^2 = k \sum_{v=0}^{k-1} \sum_{w(1)} \sum_{w(2)} c_{kw(1)+v}\, c_{kw(2)+v}\, e^{2\pi i s(w(1)-w(2))}$$

$$= k \sum_{v=0}^{k-1} \sum_{w(1)} \sum_{w(2)} c_{kw(1)+v}\, c_{kw(2)+v}\, \mathrm{Re}\,(e^{2\pi i s(w(1)-w(2))})$$

$$\leq k \sum_{v=0}^{k-1} \sum_{w(1)} \sum_{w(2)} c_{kw(1)+v}\, c_{kw(2)+v} = k \sum_{v=0}^{k-1} \Big(\sum_w c_{kw+v}\Big)^2 = k^2. \qquad \Diamond$$

A weaker condition than (Tr) is used in the next lemma. The inequality of (T1) is required only for x = 0. The following shows that under the given conditions, of course, one has absolute continuity only for some finite convolution F^{*n}.

Lemma 3.6. If $\{ s \;\; [0,1) : |p(e^{2\pi i s})| = 1 \} = \{0\}$ and $\sum_{j=0}^{k-1} \left| p(e^{2\pi i (\frac{j}{k})}) \right| = 1$, then for some $n < \infty$ sufficiently large (Tn) holds.

Proof: Let $n = 2r$, set $f_j(x) = \left| p(e^{2\pi i (\frac{x+j}{k})}) \right|^2$, and $g_r(x) = \sum_{j=0}^{k-1} f_j^r(x)$. Noting that $\varepsilon \;\; (0, 1)$ entails for some $\eta \;\; (0,1)$, $f_j(x) \;\; [0, \eta]$, for all $x \;\; [\varepsilon, 1-\varepsilon]$, one has for such x, $0 \leq g_r(x) \leq k\, \eta^r$.

It is now shown that $\varepsilon > 0$ can be chosen sufficiently small so that for all $r \geq 1$ and all $x \;\; [0,1] \setminus [\varepsilon, 1-\varepsilon]$, $0 \leq g_r(x) \leq 1$. To see this, note first that $f_j \;\; C^{\infty}(R)$ and next that $f_j(0) = 0 = f'_j(0)$ for $j = 1, 2, \ldots, k\text{-}1$ while $f_j(1) = 0 = f'_j(1)$ for $j = 0, 1, 2, \ldots, k\text{-}2$. So $g_r(0) = g_r(1) = 1$, $g'_r(0) = g'_r(1) = 0$, and $g''_r(0) =$

$g''_r(1) = r\, f''_0(0) = -\, r\, a < -\, a < 0$. Therefore if $\varepsilon > 0$ is chosen sufficiently small one has for all $r \geq 1$ and for all $0 < x < \varepsilon$, $g'_r(x) < 0$ and hence, $g_r(x) - 1 = \int_0^x g'_r(s)\, ds < 0$. Letting r be large enough so that $k\, \eta^r <$ 1 and using an analogous argument at x = 1 proves the claim. ◊

One can prove (see Deliu and Spruill (1994)) that $T_r^m \Psi^r(x)$, x [0,1], is increasing in m and that $\lim_{m\to\infty} T_r^m \Psi^r(x) = \Lambda_r(x) = \sum_{u\geq 0} \Psi^r(x+u)$ with $T_r \Lambda_r = \Lambda_r$ so that φ is in L_r if and only if

$$\int_0^1 \sum_{u\geq 0} \Psi^r(x+u)\, dx < \infty .$$

4. Necessary conditions for absolute continuity.

In this section it is proven that if F is to be absolutely continuous then p must possess certain special roots on the unit circle in the complex plane. These sets are shown to be blocking sets for a collection of points forming a directed forest; there are no paths up any tree thereof which do not encounter a zero of p. The following is basic.

Lemma 4.1. $\varphi(t) = 0$ if and only if $p(\exp\{ it / k^n\}) = 0$ for some n {1,2,…}.

Proof: Clearly, it suffices to prove that $\varphi(t) = 0$ entails $p(\exp\{ it / k^n\}) = 0$ for some n {1,2,…}. Since

$$\varphi_Y(t) = p(\exp\{itk^{-1}\})\ \varphi_Y(t/k) = \ldots = \prod_{n=1}^{m} p(\exp\{itk^{-n}\})\varphi_Y(t/k^m)$$

and φ, being a characteristic function, is continuous at 0 with $\varphi(0) = 1$, one has for sufficiently large m that

$$0 = \prod_{n=1}^{m} |\, p(\exp\{itk^{-n}\})\,|\ \ |\varphi_Y(t/k^m)| \ \geq \frac{1}{2} \prod_{n=1}^{m} |\, p(\exp\{itk^{-n}\})\,| \geq 0. \qquad ◊$$

Define

$$F_v = \{ \sum_{j=1}^{v} u(j)\, k^{-j}, \text{ where } u(j) \quad \{0,1,\ldots,k-1\} \text{ for } j=1,\ldots,v \text{ and } u(v) > 0\}$$

for v = 1, 2, 3, …. Defining y in F_m as the predecessor of x in F_{m+1} whenever $x = \frac{j}{k} + \frac{y}{k}$, for some j = 0, 1, …, k-1, and writing y > x for this relation, the set

$$T = \{ (y(1), y(2), y(3), \ldots) : y(i) > y(i+1) \text{ and } y(i) \quad F_i \text{ for all } i\}$$

is a forest with nodes $N = \quad F_i$. ($i\geq 1$)

Definition 4.2: The collection C of nodes in N is a blocking set for T if for every (y(1), y(2), y(3), …) T one has

$$C \cap (\quad_{i\geq 1} \{y(i)\}) \neq \quad .$$

Theorem 4.3. If p is associated with an absolutely continuous solution F to (CD) then the set $\{ t : p(e^{2\pi i t}) = 0\}$ contains a blocking set for T.

Proof: We shall suppose not and derive a contradiction. If the set $\{ t : p(e^{2\pi i t}) = 0\}$ does not contain a blocking set then there is a point (y(1), y(2), y(3), …) T satisfying $p(e^{2\pi i y(j)}) \neq 0$ for every j.

The first claim is that there is an index m such that $p(e^{2\pi i y(m)k^{-n}}) \neq 0$ for all $n \geq 1$; otherwise there is, for each j, an $n(j) \geq 1$ with $p(e^{2\pi i y(j)k^{-n(j)}}) = 0$. Since $p(1) = 1$ is continuous and all y(j) are in (0,1), { n(j) } must be a bounded sequence. It follows, since p is a polynomial with no more than N zeros, that there must be an infinite sequence m', an integer $v \geq 1$, and a number c   (0,1) such that $y(m')k^{-v} = c$. But all the y(m') are distinct, so this is impossible. Therefore,

$$(y'(1), y'(2), \ldots) = (y(1), y(2), y(3), \ldots, y(m), y(m)/k, y(m)/k^2, \ldots) \quad T$$

and also satisfies $p(e^{2\pi i y'(j)}) \neq 0$ for every j.

It follows from (3.1) and $p(\exp\{2\pi i k^s y(m)\}) = p(\exp\{2\pi i\, y(m-s)\})$ for $1 \leq s \leq m-1$, upon taking $t = y(m)$, that $\varphi(2\pi k^n t) = p(\exp\{2\pi i y(1)\}) \ \ldots\ p(\exp\{2\pi i y(m)\})\ \ \varphi(2\pi y(m))$ for all $n \geq m$. By construction and Lemma 4.1, none of the terms on the right of the last equality vanish and $\varphi(2\pi k^n t)$ does not tend to 0 as $n \to \infty$. By the Reimann-Lebesgue lemma and the assumption of absolute continuity a contradiction has been obtained which establishes the claim. ◊

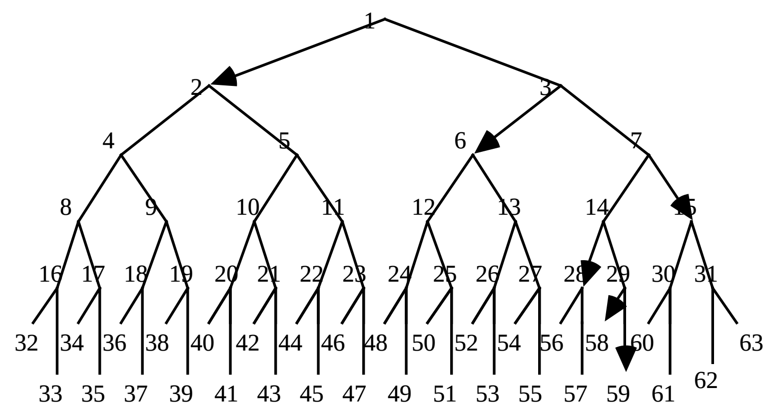
Example 4.4. Let $k = 3 = N$ and $c_0 = c_1 = c_2 = c_3$. The random variable Y is singular for consider the point $(1/3, 1/9, \ldots, 1/3^n, \ldots)$ of T. The zero set of p does not intersect $\exp\{i2\pi A\}$, where $A = \{1/3, 1/9, \ldots, 1/3^n, \ldots\}$ and hence does not form a blocking set. This follows easily from the fact that

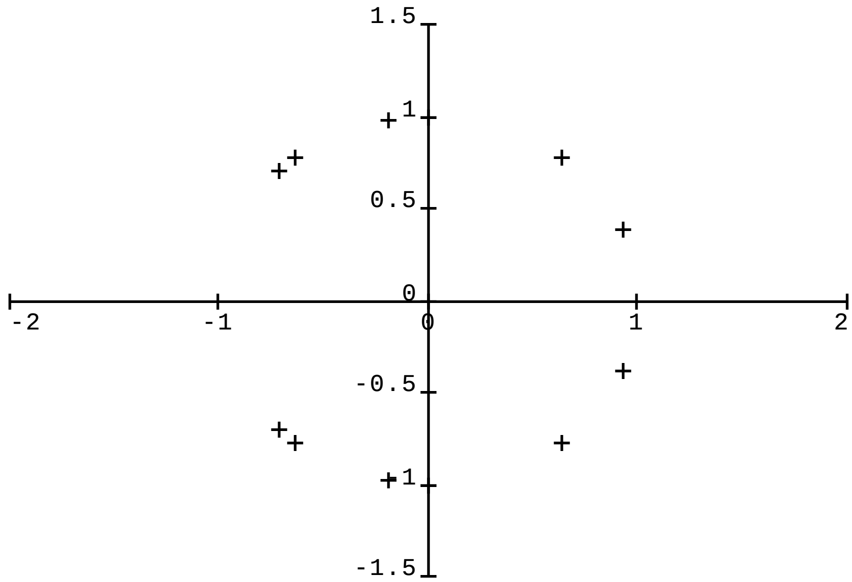
$$|\,p(z)\,|^2 = \frac{1}{16}\ |\ 1 + z + z^2 + z^3\ |^2 = \frac{1}{16}\ \left|\ \frac{1 - z^4}{1 - z}\ \right|^2 = 0$$

entails z one of the fourth roots of unity, $\{e^{2\pi i j/4} : j = 1,2,3\} = \exp\{\, i\pi B\}$, where $B = \{\ 1/2, 1, 3/2\ \}$, and $B \cap A = \emptyset$.

Example 4.5. The following graphic depiction of an initial portion of the set T and a blocking set is accompanied by the corresponding set of points in F which form a blocking set (the arrowed nodes) and a plot of the corresponding points on the unit circle together with their conjugates. There are really 12 points because the two on the imaginary axis are indistinguishable from the scale markings. Here k is 2.

1/4+0/2, 1/8+1/4+0/2, 1/16+1/8+1/4+1/2, 1/32+1/16+1/8+0/4+1/2, 1/64+1/32+1/16+0/8+1/4+0/2, 1/64+1/32+1/16+0/8+1/4+1/2





Corollary 4.6. If Y is absolutely continuous, then $\varphi(2\pi t)$ has zeros at t $\{1/k, 2/k, \ldots, (k-1)/k\}$.

Proof: Let j $\{1,2,\ldots,k-1\}$ be arbitrary. By Theorem 4.3, p must have a zero at $e^{2\pi ij/k^m}$ for some $m \geq 1$; otherwise the point $(j/k, j/k^2, \ldots)$ T is not blocked. Now use $\varphi_Y(t/k^{u-1}) = p(e^{it/k^u})\ \varphi_Y(t/k^u)$ starting with u = m and working backward to u = 2 with $t = 2\pi j$. ◊

5. Applications.

In this section can be found the complete characterizations for k = 2 of the absolutely continuous solutions to (D) for $N \leq 5$, and some odds and ends which enable extensions of the results above.

In the previous section it was seen that absolute continuity of Y entails the existence of certain roots of p on the unit circle. This implies that p can be factored in a certain way, p = q r. If, for example, the factor arising were $q(z) = 1+z$ and one knew that $p(e^{it})/q(e^{it}) = r(e^{it})$ was a non-negative definite (nnd) function then it would follow that r is the characteristic function of some random variable W and consequently that the random variable C found in the description of Y, can be written as C = U + W, where P[U = 0] = P[U=1] = 1/2 and W is independent of U. The set of absolutely continuous measures is an ideal under convolution so that Y, being the convolution of ordinary Lebesgue measure with another measure, is perforce absolutely continuous.

Definition 5.1. The function K of the real variable t is nnd if for all n, complex constants $z_1, \ldots, z_n$, and all real t(1), …, t(m), $\sum_{u=1}^{m} \sum_{v=1}^{m} z_u K(t(u) - t(v)) \bar{z}_v \geq 0$.

Proposition 5.2. (Bochner). The continuous function K is the characteristic function of a random variable if and only if it is nnd.

Lemma 5.3. The function $K(t) = p(e^{it})$ is non-negative definite (nnd) if and only if $c_j \geq 0$ for j = 0, …, N.

Proof: Easy.

Example 5.4. Consider the equation

$$f(x) = \frac{1}{8}\left(f(2x) + 3 f(2x - 1) + f(2x - 2) + 6 f(2x - 3) + f(2x - 4) + 3 f(2x-5) + f(2x - 6) \right).$$

It has a non-trivial L_1 solution because the corresponding polynomial is

$$p(z) = \frac{1}{16}(1 + 3z + z^2 + 6z^3 + z^4 + 3z^5 + z^6) = (\frac{1+z^2}{2})\ \frac{1+3z+3z^2+z^4}{8}$$

and the factor in parentheses corresponds to the characteristic function of 2U, where U is uniform on {0,1}. ◊

It is easy to see that if p is nnd, $p(x) = \frac{(1+x^m)}{2}$ r(x), and n = deg(r) < m then r is nnd automatically. More detailed results are given in Deliu and Spruill(1994).

The method of proof below in characterizing the absolutely continuous solutions involves checking the complete enumeration of all possible blocking sets. Not all blocking sets are feasible. Since p has real coefficients, conjugates are introduced and in a typical case, like example 4.5, more roots must be present than the degree allows. The set presented in Example 4.5 could not be a blocking set for any polynomial of degree less than twelve. In the following presentation only the feasible cases are presented, as determined by a search of all the possible blocking sets. The complete diagrams are available in Deliu and Spruill (1994) and were obtained by first proving the result that the maximum depth to which a search need be executed in order to discover all the possible blocking sets for an (N,k) pair is the greatest integer in N / (k-1) .

Theorem 5.5. The following are precisely the collections of coefficients resulting in a non-trivial L_1 solution to (D), or equivalently, the c's resulting in an absolutely continuous Y, for the indicated k,N pair:

k=2, N=2 - Y is absolutely continuous if and only if either (i) $c_0 + c_2 = c_1$ or (ii) $c_0 = c_2 = 1$.

k=2, N = 3 - Y is absolutely continuous if and only if either (i) $c_0 + c_2 = c_1 + c_3$ or (ii) $c_0 + c_1 = c_2 + c_3$ and $c_0 = c_2$.

k=2, N = 4 - Y is absolutely continuous if and only if either (i) $c_4 + c_2 + c_0 = c_3 + c_1$ or (ii) $c_3 = c_1$ and $c_2 = c_4 + c_0$ or (iii) $c_4 = 1 = c_0$.

k=2, N = 5 - Y is absolutely continuous if and only if the coefficients satisfy (i) $c_0 + c_2 + c_4 = c_1 + c_3 + c_5$, (ii) $c_1 - c_3 + c_5 = 0 = c_0 - c_2 + c_4$,or (iii) $c_3 = 0$, $c_2 = 0$, $c_1 - c_5 = 0 = c_0 - c_4$.

k=3, N = 3 - Y is absolutely continuous if and only if $c_1 = c_2$ and $c_2 = c_0 + c_3$.

Proof: The complete proof is given only in the case N=5 and k=2. In the other cases, the possible factors resulting from the blocking sets are presented from which one can easily fashion the remainder of the proof.

k=2, N=2 - Either (1+z) is a factor of p, which is case (i) or $p(z) = \frac{1}{2}(1+z^2)$ which is case (ii).

k=2, N = 3 - There are two cases; either (a) z+ 1 is a factor or (b) $z^2 +1$ is a factor of p.

k=2, N = 4 - Concerning blocking sets one finds there are three cases: (a) z+1 divides p, (b) $z^2 +1$ divides p, and (c) $z^4 + 1$ divides p.

k=3, N = 3 - $z^2 + z + 1$ divides p.

k=2, N = 5 (proof) -First assume that Y is absolutely continuous. Concerning blocking sets one finds there are three cases; all the others involve too many zeros of p: (a) z+1 divides p, (b) $z^2 +1$ divides p, and (c) $z^4 + 1$ divides p.

(a) This condition amounts to $p(-1) = 0$ which is just (i).
(b) Since

$$2\,p(z)/(z^2+1) = c_5 z^3 + c_4 z^2 + (c_3 - c_5) z + c_2 - c_4 + ((c_1 - c_3 + c_5) z + c_0 - c_2 + c_4) / (z^2 + 1)$$

it can be seen that (ii) is satisfied.
(c) Since

$$2\,p(z)/(z^4+1) = c_5 z + c_4 + (c_3 z^3 - c_2 z^2 + (c_1 - c_5) z + c_0 - c_4) / (z^4 + 1),$$

(iii) is satisfied.

For the converse, assume that one of the conditions (i) through (iii) holds.
(i) holds - Then Theorem. 3.5 applies and there is an L^2 solution to (D).

(ii) If (ii) holds then

$$2\,p(z)/(z^2+1) = c_5 z^3 + c_4 z^2 + (c_3 - c_5) z + c_2 - c_4 .$$

Since $c_3 - c_5 = c_1 \geq 0$ and $c_2 - c_4 = c_0 \geq 0$ it follows that $Y = 2U + W$, where U and W are independent, hence Y is absolutely continuous.
(iii) Since in this case $\deg(r) = 1 < \deg(z^4 + 1) = 4$, $Y = 4U + W$ with U and W independent so that Y is absolutely continuous. ◊

We can not go on to characterize the case $N = 6$ when $k = 2$. One can show that if Y is absolutely continuous then the factors resulting from the feasible blocking sets are $1+z$, $1+z^4$, $p(z) = (1+z^2)(1+z^4)/2$, and $1+z^2$. In every case except the last the proof can be carried to completion. In the last, however, the quotient is $c_6 z^4 + c_5 z^3 + (c_4 - c_6) z^2 + (c_3 - c_5) z + c_2 - c_4 + c_6$ and all the coefficients of this polynomial can be shown to be non-negative, except, without further conditions, $(c_4 - c_6)$. Example 5.4 is one for which this coefficient is positive. A similar phenomenon occurs with $k = 3$, except much earlier. The search for feasible blocking sets is simpler since the depth is only roughly $N/2$ rather than N; in fact, either $z^2 + z + 1$ or $z^6 + 1$ is a factor for all the absolutely continuous cases with $N \leq 7$. However, the breakdown in our technique occurs almost immediately and only the case N=3 can be solved this way.

The following are sometimes useful. Proofs are left to the reader and can be found in Deliu and Spruill (1994).

Lemma 5.5. If $p(z) = r(z^m)$ is nnd and if the corresponding random variables are Y_p and Y_r then Y_p is absolutely continuous if and only if Y_r is absolutely continuous

Example 5.6. If $p(z) = \frac{1}{16}(1 + 3z^3 + z^6 + 6z^9 + z^{12} + 3z^{15} + z^{18})$ then there is an absolutely continuous solution to (D).

Concerning reversed coefficients, the following is true.

Lemma 5.7. If $p(z) = \frac{1}{k}\sum_{n=0}^{N} c_n z^n$, $p^*(z) = \frac{1}{k}\sum_{n=0}^{N} c_{N-n} z^n$, and if Y and Y* denote the respective random variables, then Y is absolutely continuous if and only if Y* is also.

Obviously, all of the characterizations yield coefficients which when reversed still satisfy the conditions.

Example 5.8. Consider the equation

(5.1) $$f(x) = f(3x) + \frac{2}{3}(f(3x+1) + f(3x-1)) + \frac{1}{3}(f(3x+2) + f(3x-2))$$

whose "symbol" is

$$p(z) = \frac{1}{9}(1 + 2z + 3z^2 + 2z^3 + z^4)/z^2.$$

The corresponding random variable Y simply has iid C's which can take on negative values and Theorem 2.1 continues to be applicable without alteration. Therefore, (CD) has a solution which is either purely singular continuous or absolutely continuous. Similarly, Lemma 4.1 and Theorem 4.3 apply as well since the zeros of the present rational function p behave in the appropriate way. Since

$$p(z) = \frac{1}{9}(1 + z + z^2)^2/z^2,$$

it follows that $Y = U_1 + U_2 - 2\sum_{j\geq 1} 3^{-j} = U_1 + U_2 - 1$, where the U's are independent U(0,1), and hence there is an absolutely continuous solution f to (5.1), called by Daubechies and Lagarius (1991) a de Rham function . Here, f is actually continuous. ◊

6. Complements.

From example 3.4 it can be seen that the condition (T1) is not necessary for the existence of an L_1 solution to (D). How close to necessity for the existence of an L_2 solution to (D) is the sufficient condition (T2) ? If $q = kp$ and $J = \int_0^1 \ln(|q(e^{2\pi i t})|)\,dt$, then (T2) entails $J \leq \frac{1}{2}\ln k$ and if $J > \frac{1}{2}\ln k$, then there is no L_2 solution to (D). The arguments involve the random variables $\ln(|q(e^{2\pi i S k^n})|)$, $n \geq 0$, $S \sim U(0,1)$, through the behavior of the familiar averages

$$\frac{1}{m}\sum_{n=0}^{m-1} \ln(|q(e^{2\pi i S k^n})|).$$

It is well known by results from either ergodic theory (see, for example, Furstenburg(1981)) or from uniform distribution of sequences (see Kuipers and Niederreiter (1974)), that

(6.1) $$\frac{1}{m}\sum_{n=0}^{m-1} \ln(|q(e^{2\pi i s k^n})|) \to J,$$

where the convergence is for Lebesgue ae s [0,1]. Therefore, since

$$\int_0^{2\pi k^m} |\varphi(t)|\,dt = 2\pi\int_0^1 |\varphi(2\pi s)| \exp\{m(\frac{1}{m}\sum_{n=0}^{m-1}\ln(|q(e^{2\pi i s k^n})|)\}\,ds,$$

one would expect that if J were greater than 0 this integral should diverge and if it were less than 0, may or may not converge. Concerning the next result, see also Cavareta, Damen, and Micchelli (1991).

Lemma 6.1. If $r > 0$ and $J + (\frac{1}{r} - 1)\ln k > 0$ then φ L_r.

Proof: Let $0 < \varepsilon < J + (\frac{1}{r} - 1)\ln k$ and

$$B(m,\varepsilon) = \{ s \quad [0,1] : \frac{1}{m}\sum_{n=0}^{m-1}\ln(|q(e^{2\pi i s k^n})|) > \varepsilon/2 - (\frac{1}{r} - 1)\ln k \}.$$

Then since the convergence in (6.1) is also in measure, $\mu(B(m,\varepsilon)) \to 1$ as $m \to \infty$. Since

$$\int_0^{2\pi k^m} |\varphi(t)|^r \, dt \geq 2\pi \int_{B(m,\varepsilon)} |\varphi(2\pi s)|^r \exp\{ m (\ln k + \frac{1}{m} \sum_{n=0}^{m-1} \ln (| p(e^{2\pi i s k^n})|^r) \} ds$$

$$\geq 2\pi \int_{B(m,\varepsilon)} |\varphi(2\pi t)|^r \, dt \quad e^{m r \varepsilon /2}$$

and φ is a characteristic function strictly positive on an interval containing the origin, the claim follows. ◊

It is easy to see that $\sum_{j=0}^{k-1} | p(e^{2\pi i (\frac{x+j}{k})} |^r \leq 1$ entails $(\frac{1}{r} - 1) \ln k + J \leq 0$ for upon integration of the inequality (Tr) one has $k^{1-r} \int_0^1 |q(e^{2\pi i s})|^r ds \leq 1$. Using Jensen's inequality,

$$1 \geq \exp\{ (1-r)\ln k + \ln(\int_0^1 |q(e^{2\pi i s})|^r ds) \} \geq \exp\{ (1-r)\ln k + r \int_0^1 \ln(|q(e^{2\pi i s})|) \, ds \}.$$

Lemma 6.2. If $J > \frac{1}{2} \ln k$ then there is no L_2 solution to (D).

Proof: If there were an L_2 solution to (D) one would have by compact support that $f \quad L_1 \cap L_2$, and then the Fourier-Plancherel transform which coincides with the Fourier transform φ would entail $\varphi \quad L_2$. Therefore, there is no L_2 solution to (D). ◊

The role of the integral J in determining the absolute continuity of Y seems unclear. This can be seen from an application of Jensen's formula (see Rudin(1966) Theorem 15.18) and the results on blocking sets above. Jensen's formula says that if $p(0) \neq 0$ and $z_1, \ldots, z_M$ are the zeros of p(z) in the closed unit disk listed according to their multiplicities, then $\int_0^1 \ln(| p(e^{2\pi i t}) |) \, dt = \ln(|p(0)|) - \sum_{j=1}^{M} \ln(|z_j|)$. Rewriting the condition of Lemma 6.1 in these terms, using Jensen's formula and

$$c_0 / k = \prod_{j=1}^{N} |z_j| / \prod_{j=1}^{N} |1 - z_j| = |p(0)|,$$

we have the following, where D is the set of zeros, z, of p in $|z| \leq 1$ and D^c is the set of zeros of p outside the unit disk.

Theorem 6.3. A necessary condition for $\varphi \quad L_r$ is $k^{1/r} \leq \prod_{z \quad D} |1-z| \prod_{z \quad D^c} |1/z-1|$.

The condition is obviously not sufficient. In the case of the Cantor random variable Y, for example, k = 3, $p(z) = (1+z^2)/2$, and although $\sqrt{3} \leq 2$ there is no L_2 solution; Y is singular.

The methods so far, employing characteristic functions, must necessarily fall short of a complete description of those equations (D) possessing a non-trivial L_1 solution; continuous densities are known (we're not claiming they satisfy (D)) which have characteristic functions not in L_1 and even the uniform density, which is continuous on its support, fails to have a characteristic function in L_1. Perhaps a preferable method is to pursue a course indicated by the work of Strichartz (1990) or, proceeding more directly in terms of the measure itself, adapt a result like Theorem 1.10 of Garsia to geometric intervals. An analysis could perhaps be based on the following particularly appealing fact whose proof is immediate from the work here and in Van der Vaart (1967).

Theorem 6.4. The random variable Y is absolutely continuous if and only if

$$\int_0^{N/(k-1)} \lim_{T\to\infty} h_T(s)\, ds = 1, \text{ where } h_T(s) = (2\pi)^{-1} \int_{-T}^{T} [1 - \frac{|t|}{T}]\, e^{-ist}\, \varphi(t)\, dt.$$

Effective in some simple situations is the following.

Corollary 6.5. If $E[\,|Y - x|^{-2}\,] < \infty$ for a.e. x in $[0, \frac{N}{k-1}]$ then Y is singular.

Proof: Since $(2\pi)^{-1} \int_{-T}^{T} [1 - \frac{|t|}{T}]\, e^{-ixt}\, \varphi(t)\, dt = (2\pi T)^{-1} \int_0^T \int_{-t}^{t} \varphi_{Y-x}(s)\, ds\, dt$ and

$$\int_{-t}^{t} e^{-ixs}\, \varphi_Y(s)\, ds = E[\frac{2 \sin(t(Y-x))}{Y-x}],$$

if for a.e. x, $E[\,|Y-x|^{-2}] < \infty$, then $(2\pi T)^{-1} \int_0^T \int_{-t}^{t} \varphi_{Y-x}(s)\, ds\, dt = (\pi T)^{-1} E[\frac{1 - \cos(T(Y-x))}{(Y-x)^2}] \to 0$ as $T \to \infty$. ◊

Example 6.6. If $p(z) = \frac{1}{2}(1 + z^2)$ and k = 3 then Y is the ordinary Cantor random variable whose singularity could easily be determined from Theorem 4.3, but is also immediate from Cor. 6.5 since for x in the deleted intervals, $|Y-x|$ is bounded away from zero. Clearly $E[\,|Y - x|^{-2}\,] < \infty$ for all such x, a set of measure 1. ◊

References

1. Berger, M. and Wang, Y. (1992). Multidimensional two scale dilation equations. In *Wavelets - A Tutorial in Theory and Applications*, C. K. Chui, Ed., Academic Press.
2. Cavareta, A.S., Damen, W., and Micchelli, C.H. (1991). Stationary subdivision, *Mem. Amer. Math. Soc.*, 93, pp 1--186.
3. Cohen, A. and Conze, J.P. (1992). Régularité des bases d'ondelettes et mesures ergodiques. *Revista Matemática Iberoamericana*, 8 351-365.
4. Cohen, A. and Sun, Q. (1993). An arithmetic characterizationof the conjugate quadrature filters associated to orthogonal wavelet bases. *SIAM J. Math. Anal.* 24 1355-1360.
5. Colella, D. and Heil, C. (1992). The characterization of continuous four-coefficient scaling functions and wavelets, *IEEE Transactions on Information Theory* 38 2 876-881.
6. Colella, D. and Heil, C. (1994). Characterizations of scaling functions I. continuous solutions, *SIAM J. Matrix Analysis Appl*. 15 1-23.
7. Conze, Jeanne-Pierre and Raugi, Albert (1990). Fonctions harmoniques pur un operateur de transition et applications. 118 273-310, *Bull. Math. Soc. France*.
8. Daubechies, I. and Lagarius, J. C. (1991). Two scale difference equations, I. Existence and global regularity of solutions, *SIAM J. Math. Ann.* 22 1388-1410.
9. Deliu and Spruill (1994). Dilation equations and absolute continuity of random expansions. School of Mathematics Technical Rept. # 103194-025, Ga. Tech.
10. Dubins, L. and Freedman, D. (1966). Invariant probabilities for certain Markov processes, *Ann. Math. Stat*. 32 837-848.
11. Eirola, T. (1992). Sobolev characterization of solutions of dilation equations, *SIAM Jnl Math Anal* 4 1015-1030.

12. Erdös, P. (1940). On the smoothness properties of a family of Bernoulli convolutions. *Am. J. Math*. 62 180-186.
13. Furstenburg, H. (1981). *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, Princeton, N.J.
14. Garsia, A. (1962). Arithmetic properties of Bernoulli convolutions. *Trans. Amer. Math. Soc.* 102 409-432.
15. Heil, C. (1992). Methods of solving dilation equations. In *Probabilistic and Stochastic Methods in Analysis*, J. S. Byrnes et. al. eds., Kluwer, The Netherlands.
16. Jesson, B. and Wintner, A. (1935). Distribution functions and the Riemann Zeta function. *Trans. Am. Math Soc*.
17. Jia, Rong-Qing and Wang , J. (1993). Stability and linear independence associated with wavelet decompositions. *Proceedings of the American Mathematical Society* 117 4 1115-1124.
18. Lau, K.-S. and Wang, J. (1994). Characterization of L_p solutions for the two scale dilation equations. Preprint.
19. Kawata (1972). *Fourier Series in Probability Theory*, Academic Press, New York.
20. Kershner, R. and Wintner, A. (1935). On symmetric Bernoulli convolutions. 1935 Am. J. Math.
21. Kuipers, L. and Niederreiter, H. (1974). *Uniform Distribution of Sequences*, Wiley Interscience, New York.
22. Micchelli, C.A. and Prautzsch,H. (1987). Refinement and subdivision for spaces of integer translates of a compactly supported function, *IBM Research Report* RC 13175 (#58951)
23. Rudin, W. (1966). *Real and Complex Analysis*. McGraw Hill, New York.
24. Strang, G. (1989).Wavelets and dilation equations: a brief introduction, *SIAM Review* 31 614-627.
25. Strichartz, R. S. (1990). Self-similar measures and their Fourier transforms, *Indiana University Mathematics Journal* 39 3 797-817.
26. Van der Vaart, H. R. (1967). Determining the absolutely continuous component of a probability distribution from its Fourier-Steiltjes transform. *Arkiv för Matematik* 7331-342.
27. Wintner, A. (1947). *The Fourier Transforms of Probability Distributions*. Edwards Bros. Ann Arbor Mich.