Uninterupted tails in coin tossing A question from a former student reads:

Hello Dr. Spruill,

I was a student in your statistics class last Summer. I may have been paying more attention to the nice weather than academics, because I am unable to answer the following question that my friend (also a GT student) and I have been discussing:

Is it possible to "flip a coin" forever and always get 'tails'? My friend and I have defined this as a thought experiment, as one clearly cannot flip a coin ad infinitum. Additionally, it shall be 'possible' to get either heads or tails with this coin - ie. nothing prevents either from occurring. We are hesitant to use the term 'probability' because we don't know the implications of its definition when dealing with infinity.

If you have the time, I would greatly appreciate any explanations or background information on the territory we are covering. I realize that this is a potentially sticky area; one math professor already backed out of an answer, citing that he was "not a statistics guy". Also, please let me know if I need to clarify any areas of the question.

Thank you for your time.

Tyler Graff

Solution: The answer is that the probability of such an event is 0.

Dear Tyler,

Even if you had paid attention more to academics than the weather you would not have had this question answered thoroughly in the probability coverage for that course. This kind of question is more advanced and you will probably not encounter a careful answer until you take a measure-theoretic graduate level probability course.

Nevertheless, we did cover some things that enable an answer. The axioms of probability are three and concern the function P which maps subsets A (called events) of the outcome space $\mathcal{S}$ to the real numbers. If the function P satisfies the following properties then it is a probability. Note that the third axiom deals with such "infinite" events.

1. For each event A, one has $P(A) \geq 0$.
2. $P(\mathcal{S}) = 1$
3. If $A_1, A_2, A_3, \ldots$ are events then $\cup_{j\geq 1} A_j$ is an event and if for all $i \neq j$ $A_i \cap A_j = \emptyset$ then

$$P(\cup_{j\geq 1} A_j) = \sum_{j\geq 1} P(A_j). \tag{1}$$

We used this for many things along the way, but perhaps you will recall an example we solved:

Example 1 *If I toss a coin with probability p of heads on each toss and the outcomes are independent, what is the probability the first head occurs on an even toss?*

Solution: Let X be the number of tosses till the first head. Then X has a geometric distribution and for $k = 1, 2, \ldots$

$$P[X = k] = (1-p)^{k-1} p. \tag{2}$$

Thus the probability we seek is, letting A_j be the event $X = j$, and noting that for $u \neq v, A_u \cap A_v = \emptyset$ (since X can not be both values- the first head was either on the u^{th} toss, the v^{th} toss, or neither of these, but never more than one),

$$P[\text{ first head occurs on even }] = P[\cup_{i\geq 1} A_{2i}] = \sum_{i\geq 1} P(A_{2i}).$$

Since $P(A_{2i}) = q^{2i-1}p$, one has

$$P[\text{ first head occurs on even }] = \sum_{i\geq 1} q^{2i-1}p = pq\sum_{j\geq 0}(q^2)^j = \frac{pq}{1-q^2} = \frac{q}{1+q}.$$

For a fair coin this is 1/3.

You may ask what this has to do with your original question. It is just an illustration that we really did compute probabilities of such "infinite" events. Proceeding to your question now, we have:

Example 2 *The probability of all tails in an infinite number of tosses is 0.*

Solution: The event whose probability we seek is $\cap_{j\geq 1}B_j$, where B_j is the event of a tail on toss j. We have

$$P[\text{ all tails }] = P[\cap_{j\geq 1}B_j]$$

and that for every $n = 1, 2, \ldots$ one has

$$\cap_{j\geq 1}B_j \subset \cap_{j=1}^{n}B_j.$$

From a theorem we proved in class, it follows that for every n

$$P[\cap_{j\geq 1}B_j] \leq P[\cap_{j=1}^{n}B_j].$$

Let q be the probability of a tail on a single toss ($q \in (0, 1)$ under your assumptions). If the coin is tossed so the outcomes are independent then $P[\cap_{j=1}^{n}B_j] = q^n$ so for every n

$$P[\text{ all tails }] \leq q^n.$$

It follows by letting $n \to \infty$ that

$$P[\text{ all tails }] = 0.$$

Problem 1 *What is the probability of getting a finite number of "ones" in tossing an ordinary six sided die ad infinitum.*