Determinacy of Means of Order Statistics *

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August 3, 2009

1 Abstract

If $X_1, X_2, \ldots$ are independent random variables distributed as X and if $E[|X|] < \infty$ then the sequence $\rho_n = E[X_1 \vee X_2 \vee \ldots \vee X_n]$ of maximal moments determines the probability distribution of X. Exploiting the fact that a collection of functions is fundamental if and only if a corresponding moment problem is determinate yields simple proofs of the claim above, a theorem of Pollak on the determinacy of more general sequences of means of order statistics, and that a subsequence n(i) of the maximal moments determines the distribution if and only if $\sum_{i\geq 1} n^{-1}(i) = \infty$.

2 Introduction

In this paper the relationship between the linear spans of collections of functions and the measures they determine is exploited to characterize which maximal moments determine a probability distribution.

Müntz (see Natanson [8]) sought conditions, in various spaces, on the powers $p(i)$ which render the collection of functions $\{t^{p(i)}\}_{i\geq 1}$ a fundamental set. One of his theorems pertains to $C[0,1]$ and says that the collection $\{t^{n(i)}\}_{i\geq 1}, 0 \leq n(1) < n(2) < \ldots$, is fundamental (has dense linear span) in $C[0,1]$ if and only if $n(1) = 0$ and $\sum_{i\geq 1} n^{-1}(i) = \infty$. This question of which collections$\{t^{n(i)}\}_{i\geq 1}$ are fundamental can be recast, as can be seen below, as one of determinacy of a corresponding class of moments. The polynomial moments, or simply moments, of a measure μ are $M_n(\mu) = \int t^n d\mu(t)$ and in this alternative form the answer to the question of which collections $\{t^{n(i)}\}_{i\geq 1}$ are fundamental is: the subsequence $\{t^{n(i)}\}_{i\geq 1}$ is fundamental exactly when the collection of numbers $\{M_{n(i)}(\mu)\}_{i\geq 1}$ determines the Borel signed measures μ on $[0,1]$.

The determinacy of measures by their moments has been thoroughly studied for polynomial powers (see Akhiezer [1]) with definitive results on compact intervals. When the measures are on the whole line results are not as sharp; Carleman's condition (see Shohat and Tamarkin [10]) states

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[*] Following the link one can find an electronic version of the paper in Sankhya Ser. A 57 (1995), but it contains several typographical errors not in the original printed version of the journal. It is hoped this one is free of such.

that the probability distribution F is uniquely determined by its moment sequence $\{M_n\}_{n\geq 1}$, if it exists, whenever $\sum_{n\geq 1} M_{2n}^{-1/2n}$ diverges and examples (see Feller [2]) show that there are distinct probability distributions with the same moment sequence when this condition fails. In contrast, consider the maximal moment sequence $\{\rho_n\}_{n\geq 1}, \rho_n = E[X_1 \vee X_2 \vee \ldots \vee X_n]$, which exists if X is in L_1. That the sequence $\{\rho_n\}_{n\geq 1}$ determines X's probability distribution function F was first pointed out by Hoeffding [5]. M. Pollak [9] proved the following stronger result. Let $X_{1:n} \ldots X_{n:n}$ denote order statistics from $X_1, \ldots, X_n$, independently and identically distributed random variables with cdf F, and let $\rho_{j:n} = E[X_{j:n}]$ for $n \geq 1$ and $j \in \{1, 2, \ldots, n\}$. If there is a random variable Y with the property that for each n there exists a $j(n)$ with $E[X_{j(n):n}] = E[Y_{j(n):n}]$, then $E[X_{j:n}] = E[Y_{j:n}]$ for all $j = 1, \ldots, n$ and all $n \geq 1$ and this implies that X and Y have the same probability distribution. Thus for any sequence $\{j(n)\}_{n\geq 1}$ such that $j(n) \in \{1, 2, \ldots, n\}$ for all n, the sequence $\{\rho_{j(n):n}\}_{n\geq 1}$ determines F. Pollack's result is proven below quite simply by invoking fundamental sequences and the correspondence between fundamental sets and determinacy of moment problems cited above.

Although Pollak's method, which uses induction on n, proves that the sequence $\{\rho_n\}_{n\geq 1}$ determines the cdf F, it can not be used to prove that the sequence $\{\rho_{n(i)}\}_{i\geq 1}$ determines F if and only if $\sum_{i\geq 1} n^{-1}(i) = \infty$. In the method employed here to answer this question, collections of functions other than polynomial powers can also be investigated. For example, as will be seen below, this method shows that no proper subset of the Chebyshev polynomials of the first kind has dense linear span; one interesting consequence is that if C_n is the nth Chebyshev polynomial on $[0, 1]$ and if $k > 1$ is an arbitrary integer, then there are distinct probability measures $P_i, i = 1, 2$, on $[0, 1]$, absolutely continuous with respect to Lebesgue measure, for which $\int C_n(t)dP_1(t) = \int C_n(t)dP_2(t)$ for every $n \in \{0, 1, 2, \ldots\}$ except $n = k$.

Utilization of the relationship between determinacy and fundamental sets of polynomials has been made by previous researchers. For example, Huang [6] uses it to prove the if part of our Theorem 4. See Huang [7] for a review article on this topic and see Hill and Spruill [4] and [3] for the use of maximal moments to investigate weak convergence.

3 Determinacy of Generalized Moments and Fundamental Systems

For a signed measure $\nu = \nu^+ - \nu^-$ on the measure space $(\Omega, \mathcal{A})$, denote the total variation measure by $|\nu| = \nu^+ + \nu^-$ and the total variation norm of ν by $\|\nu\| = |\nu|(\Omega)$. The space BV is the normed linear space of signed measures ν with $|\nu|(\Omega) < \infty$. The generalized moments of a signed measure ν are $\{\int \phi_\gamma d\nu\}_{\gamma\in\Gamma}$, where $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is a family of real valued functions on Ω measurable from $\mathcal{A}$ to the Borel sets $\mathcal{B}$ in R^1.

Definition 1 *The generalized moments $\{\int \phi_\gamma d\nu\}_{\gamma\in\Gamma}$ determine the signed measures in $C \subset BV$ on $\mathcal{A}$ if for any two signed measures $\nu_i \in C$, $\int \phi_\gamma d\nu_1 = \int \phi_\gamma d\nu_2$ for all $\gamma \in \Gamma$ entails $\nu_1 = \nu_2$.*

Denoting by $\mathcal{A}$ the Baire σ-field of subsets of Ω, the minimal σ-field with respect to which all bounded real valued continuous functions on Ω are measurable with respect to $\mathcal{B}$, the Riesz representation theorem shows that if Ω is a compact Hausdorff space and ν_1 and ν_2 are two signed measures on $(\Omega, \mathcal{A})$, then $\nu_1 = \nu_2$ if and only if $\int f d\nu_1 = \int f d\nu_2$ for all $f \in C(\Omega)$.

The family $\{\phi_\gamma\}_{\gamma\in\Gamma}$ of functions in a normed linear space D is fundamental if for each $f \in D$ and $\epsilon > 0$ there is a finite linear combination $\sum_{j=1}^{n} a_j\phi_{\gamma(j)}$ of elements of the family for which

$$\|f - \sum_{j=1}^{n} a_j\phi_{\gamma(j)}\| < \epsilon.$$

Theorem 1 *If Ω is a compact Hausdorff space, then the family $\{\phi_\gamma\}_{\gamma\in\Gamma}$ of functions in $C(\Omega)$ determines the signed measures ν in BV if and only if it is fundamental in $C(\Omega)$.*

Proof: Let M denote the closed linear span of $\{\phi_\gamma\}_{\gamma\in\Gamma}$ in $C(\Omega)$ with the supremum norm. By the Hahn-Banach theorem, if $M \neq C$ then there is an $f \notin M$, and a bounded linear functional ν such that $\|\nu\| = 1, \nu$ vanishes on M, and $\int f d\nu = \inf_{m\in M} \|f - m\| = h > 0$. Writing $\nu = \nu^+ - \nu^-$, one has for all $\gamma \in \Gamma$

$$\int \phi_\gamma(x)d\nu^+(x) = \int \phi_\gamma(x)d\nu^-(x).$$

Since

$$\int f(x)d\nu^+(x) \neq \int f(x)d\nu^-(x),$$

the family does not determine the measures.

Now suppose $M = C(\Omega)$. Let ν_1 and ν_2 be two signed measures satisfying $\int \phi_\gamma d\nu_1 = \int \phi_\gamma d\nu_2$ for all $\gamma \in \Gamma$ and let $f \in C(\Omega)$ be arbitrary. To show that $\int f d\nu_1 = \int f d\nu_2$ let $\epsilon > 0, K = \max\{\|\nu_1\|, \|\nu_2\|\}$, and $g = \sum_{j=1}^{n} aj\phi_{\gamma(j)}$ be such that $\|f - g\| < \epsilon/2K$. Then

$$|\int f d\nu_1 - \int f d\nu_2| = |\int f d\nu_1 - \int g d\nu_1 - (\int f d\nu_2 - \int g d\nu_2) + \int g d\nu_1 - \int g d\nu_2| < \epsilon.$$

Since ϵ and f were arbitrary the two measures agree. □

Since the polynomials are dense in $C[0, 1]$, it follows that $\{t^n\}_{n\geq 0}$ determines the signed Borel measures on $[0, 1]$. It is clear that if for each n, Q_n is a polynomial of degree n, then for each polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$ there is a unique sequence of scalars b_i such that

$$p(x) = \sum_{i=0}^{n} b_i Q_i(x). \tag{1}$$

Therefore, any sequence of polynomials $\{Q_n\}_{n\geq 0}$, where Q_n is of degree n, determines the signed Borel measures on $[0, 1]$.

Let $p \in [1, \infty]$ and denote by $AC(p)$ (resp. $PAC(p)$) the set of signed measures (resp. probability measures) on the measure space $([0, 1], \mathcal{B})$ which are absolutely continuous with respect to Lebesgue measure and have a Radon-Nikodym derivative in $L_p[0, 1] = L_p$. Let q be conjugate to p: $p^{-1}+q^{-1} = 1$. Entirely analogous arguments to those in Theorem 1 can be used to establish the next two facts.

Lemma 1 *If $1 \leq p \leq \infty$ and the set of functions $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is fundamental in L_p then $\{\phi_\gamma\}_{\gamma\in\Gamma}$ determines the measures in $AC(q)$.*

Lemma 2 *If $1 < p \leq \infty$ and the set of functions $\{\phi_\gamma\}_{\gamma\in\Gamma}$ containing 1 determines the measures in $PAC(p)$ then $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is fundamental in L_q.*

These can be conveniently stated as follows.

Theorem 2 *If $1 < p \leq \infty$ then the set of functions $\{\phi_\gamma\}_{\gamma\in\Gamma}$ containing 1, determines the measures in $PAC(p)$ if and only if $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is fundamental in L_q.*

4 Application to Order Statistics

If $X_1, \ldots, X_n$ are independently and identically distributed random variables with $E(|X_1|) < \infty$ and right continuous probability distribution F, then setting for all $t \in (0,1)$

$$F^{-1}(t) = \inf\{x : F(x) \geq t\},$$

the left-continuous inverse of F, one has the following standard result.

Proposition 1

$$\rho_{k:n} = \int_0^1 \frac{n!}{(k-1)!(n-k)!} u^{k-1}(1-u)^{n-k} F^{-1}(u) du. \tag{2}$$

Proof: If $U_1, \ldots, U_n$ are iid uniform on $(0,1)$ then $F^{-1}(U_{1:n}), \ldots, F^{-1}(U_{n:n})$ have the same probability distribution as $X_{1:n}, \ldots, X_{n:n}$. The claim is established since F^{-1} is monotone and the probability density function of $U_{k:n}$ at $u \in (0,1)$ is $\frac{n!}{(k-1)!(n-k)!} u^{k-1}(1-u)^{n-k}$. □

Define the signed measure μ by $\mu([0,t]) = \int_0^t F^{-1}(s)ds$. Since

$$E[|X|] = \int_0^1 |F^{-1}(t)|dt < \infty$$

one has $|\mu| = \mu^+([0,1]) + \mu^-([0,1]) < \infty$ and μ is of bounded variation and absolutely continuous with respect to Lebesgue measure.

Theorem 3 *(Pollak) For any sequence $\{j(n)\}_{n\geq 0}$ such that $j(n) \in \{0,1,2,\ldots,n\}$ for all n, the sequence $\{\rho_{j(n):n}\}_{n\geq 0}$ determines F.*

Proof: For any sequence $\{j(n)\}_{n\geq 0}$ such that $j(n) \in \{0,1,2,\ldots,n\}$ for all n, the sequence $t^{j(n)}(1-t)^{n-j(n)} = Q_n(t)$ is a sequence of strictly increasing degree so that (1) shows it to be fundamental in $C[0,1]$. It follows that the density, F^{-1}, of μ is determined a.e. m. By monotonicity and left continuity it follows that F^{-1}, and hence F (right-continuous), is determined everywhere. □

Is it possible that some subsequence $\{\rho_{n(k)}\}_{k\geq 0}$ determines the probability distribution? Which subsequences do so? It is an immediate consequence of Theorem 1 and theorems of Müntz that the sequence $\{t^{n(i)}\}_{i\geq 0}$, with $n(0) = 0$, determines the signed measures on $[0,1]$ if and only if the sum in (3) below diverges and this is related, but not equivalent, to the problem at hand (determinacy of $\{\rho_{n(k)}\}_{k\geq 0}$). It can be seen from (2) that the determinacy of the sequence $\{\rho_{n(k)}\}_{k\geq 0}$ applies to a class of signed measures which differs from the collection of all signed measures on $[0,1]$; each member of our class is absolutely continuous with respect to Lebesgue measure and has a density which is monotonic non decreasing. Since our class is much smaller there is something to prove; it seems conceivable that the full Müntz class may not be required.

Theorem 4 *The sequence* $\{\rho_{n(i)}\}_{i\geq 1}$, *determines* F *if and only if*

$$\sum_{i\geq 1} \frac{1}{n(i)} \tag{3}$$

diverges.

Proof: It is first shown that if the sum in (3) diverges then the sequence $\{\rho_{n(i)}\}_{i\geq 1}$, determines F. Let F_1 and F_2 satisfy

$$\int_0^1 |F_j^{-1}(t)|dt < \infty \text{ for } j = 1, 2$$

and have the same ρ-subsequences so that $\int_0^1 t^{n(i)-1}F_1^{-1}(t)dt = \int_0^1 t^{n(i)-1}F_2^{-1}(t)dt$ for all $i \geq 1$. Introduce the functions h_j in L_1 defined by $h_j(t) = t^{n(1)-1}F_j^{-1}(t)$ and define the corresponding signed measures μ_j for $j = 1, 2$, where

$$\mu_j([0, t]) = \int I_{[0,t]}(x)h_j(x)dx.$$

One has for all $i \geq 1$ that

$$\int_0^1 t^{n(i)-n(1)}d\mu_1(t) = \int_0^1 t^{n(i)-n(1)}d\mu_2(t).$$

By Müntz's theorem the collection $\{t^{n(i)-n(1)}\}_{i\geq 1}$ has dense linear span in $C[0, 1]$ so that by Theorem 1, $\mu_1 = \mu_2$. Taking Radon-Nikodym derivatives shows $F_1^{-1}(t) = F_2^{-1}(t)$ for almost every $t \in (0, 1)$. By monotonicity and left continuity it follows that $F_1 = F_2$.

Now suppose that (3) converges; the existence of two distinct random variables with the same sequences $\{\rho_{n(i)}\}_{i\geq 0}$ is established. By another of Müntz's theorems, if the sum (3) converges then the sequence $\{1, t^{n(0)}, t^{n(1)}, \ldots\}$ does not have dense linear span in $L_2[0, 1]$. By Theorem 2 there are distinct probability measures μ_1 and μ_2, both absolutely continuous with respect to Lebesgue measure, with the same sequences $\{\int_0^1 t^{n(i)}d\mu_j\}_{i\geq 0}$, where $n(0) = 0$, for $j = 1, 2$. Writing $f_j = \frac{d\mu_j}{dm}$ and integrating by parts, one has equality for $j = 1, 2$ of the expressions on the right

$$\int_0^1 t^{n(i)}f_j(t)dt = t^{n(i)}\mu_j([0, t])|_0^1 - n(i)\int_0^1 t^{n(i)-1}\mu_j([0, t])dt = 1 - n(i)\int_0^1 t^{n(i)-1}\mu_j([0, t])dt$$

for every $i \geq 1$. Letting G_j be the cumulative distribution function corresponding to μ_j, and setting $F_j = G_j^{-1}$ shows that the random variables X_j whose cdf's are F_j have the same sequences $\{\rho_{n(i)}\}_{i\geq 0}$ but $F_1 \neq F_2$. □

The random variables constructed for the case in which (3) converges take values in $[0, 1]$ and are absolutely continuous with respect to Lebesgue measure.

Example 1 *The Weibull Distributions. If* X *is a random variable having the sequence*

$$E[\min\{X_1, \ldots, X_{n(i)}\}] = Kn(i)^{-1/\beta}$$

for some $\beta > 0$, *where* $X_1, \ldots, X_n$ *are iid as* X, *and if* $\sum_{i\geq 1} n^{-1}(i) = \infty$, *then* X *has the Weibull distribution with parameters* α, β, *where*

$$\alpha = \frac{K}{\Gamma(1 + 1/\beta)}.$$

Precisely which subcollections of the functions $\{t^j(1-t)^{n-j})\}_{j=0}^n$, for $n \geq 0$, are fundamental in $C[0,1]$ is unknown. Taking $j(k) = k$ the set $\{t^k(1-t)^k\}_{k\geq 0}$ is not fundamental in $C[0,1]$ since it fails to distinguish the two probability measures $U[0,1/2]$ and $U[1/2,1]$. However, this is just the sequence which picks for j, in $t^j(1-t)^{n-j}$, the middle member on the even integers $n(k) = 2k$ and $\sum_{k\geq 1}(2k)^{-1} = \infty$; so at one end, the maximal moments, there are enough functions, but in the middle there are not enough.

The remark at the end of section 1 is easily verified. By (1) any infinite sequence of orthonormal polynomials with respect to any Borel measure on $[0,1]$ determines the signed Borel measures on $[0,1]$ including of course the Chebyshev polynomials C_n on $[0,1]$, but Müntz's theorems do not hold for any such collection; no subcollection of a set orthogonal with respect to a weight function in L_1 can be fundamental in $C[0,1]$. This is not the only way to verify the claim at the end of section 1. Using the orthogonality alone one can easily prove the stronger result: if $\{Q_n\}_{n\geq 0}$ are orthonormal polynomials with respect to a Borel measure μ on $[0,1]$ and if $1 < n(1) < n(2) < \ldots < n(m)$ then there are probability measures $P_i, i = 1,2$, on $[0,1]$, both absolutely continuous with respect to m, such that for $n \notin \{n(1),\ldots n(m)\}$

$$\int Q_n dP_1 = \int Q_n dP_2$$

and for $n \in \{n(1),\ldots,n(m)\}$

$$\int Q_n dP_1 > \int Q_n dP_2.$$

5 Acknowledgement

I was introduced to problems of determinacy of maximal moments by Ted Hill; a method of maximal moments for investigating convergence of random variables can be found in Hill and Spruill [4].

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